‘

CATEGORICITY FOR ABSTRACT CLASSES WITH AMALGAMATION

SAHARON SHELAH

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Department of Mathematics
Rutgers University
New Brunswick, NJ USA

ABSTRACT. Let $\mathfrak{K}$ be an abstract elementary class with amalgamation, and Lowenheim Skolem number $LS(\mathfrak{K})$. We prove that for a suitable Hanf number χ_0 if $\chi_0 < \lambda_0 \leq \lambda_1$, and $\mathfrak{K}$ is categorical in λ_1^+ then it is categorical in λ_0.

Typeset by $\mathcal{AMS}$-TEX

Annotated Content

I§0 <u>Introduction</u>

[We review background and some definitions and theorems on abstract elementary classes.]

I§1 <u>The Framework</u>

[We define types, stability in λ, $\mathscr{S}(M)$ and E_μ: equivalence relations on types all whose restrictions to models of cardinality $\leq \mu$ are equal. We recall that categoricity in λ implies stability in $\mu \in [LS(\mathfrak{K}), \lambda)$.]

I§2 <u>Variant of Saturation</u>

[We define $<^\ell_{\mu,\alpha}$ and "N is (μ, κ)-saturated over M" and show universality and uniqueness.]

I§3 <u>Splitting</u>

[We note that stability in μ implies that there are not so many μ-splittings.]

I§4 <u>Indiscernibility and E.M. models</u>

[We define strong splitting and dividing, and connect them to the order property and unstability.]

I§5 <u>Rank and Superstability</u>

[We define one variant of superstability; in particular categoricity implies it.]

I§6 <u>Existence of many non-splitting</u>

[We prove (e.g. for $\mathfrak{K}$ categorical in $\lambda = \operatorname{cf}(\lambda)$) that if $M_0 <^1_{\mu,\kappa} M_1 \leq_{\mathfrak{K}} N \in \mathfrak{K}_{<\lambda}$ and $p \in \mathscr{S}(M)$ does not μ-split over M_0, then p can be extended to $q \in \mathscr{S}(N)$ which does not μ-split over M_0.
(Note: up to E_μ-equivalence the extension is unique). Secondly, if $\langle M_i : i \leq \delta \rangle$ is $\leq^1_{\mu,\kappa}$-increasing continuous in K_μ and $p \in \mathscr{S}(M_\delta)$ then for some i we have: p does not μ-split over M_i.]

I§7 <u>More on Splitting</u>

[We connect non-splitting to rank and to dividing.]

II§8 <u>Existence of nice Φ</u>

[We try to successively extend the Φ we use which is proper for linear orders such that we have as many definable automorphisms as possible. We also relook at omitting types theorems over larger model (so only restrictions will appear).]

II§9 <u>Small Pieces are Enough and Categoricity</u>

> [The main claim is that for some not too large χ, if $p_1, p_2 \in \mathscr{S}(M)$ are
> E_χ-equivalent, $\|M\| < \lambda$ where K is categorical in λ we have $p_1 \, E_\chi \, p_2 \Leftrightarrow$
> $p_1 = p_2$.
> Lastly, we derive that categoricity is downward closed for successor cardinals
> large enough above $\mathrm{LS}(\mathfrak{K})$.]

§0 Introduction

We try to find something on

$$\mathrm{Cat}_K = \{\lambda : K \text{ categorical in } \lambda\}$$

for $\mathfrak{K}$ an abstract elementary class with amalgamation (see 0.1 below).
The Los conjecture = Morley theorem deals with the case where K is the class
of models of a countable first order theory T. See [Sh:c] for more on first order
theories. What for T a theory in an infinitary language? (For a theory T, K is
the class $K_T = \{M : M \models T\}$ we may write Cat_T instead of $\mathrm{Cat}_{K_T} = \mathrm{Cat}_K$).
Keisler gets what can be gotten from Morley's proof on $\psi \in L_{\aleph_1, \aleph_0}$. Then see
[Sh 48] on categoricity in $\aleph_1$ for $\psi \in L_{\aleph_1, \aleph_0}$ and even $\psi \in L_{\aleph_1, \aleph_0}(Q)$, and [Sh
87a], [Sh 87b] on the behaviour in the $\aleph_n$'s. Makkai Shelah [MaSh 285] proved:
if $T \subseteq L_{\kappa, \aleph_0}, \kappa$ a compact cardinal then $\mathrm{Cat}_T \cap \{\mu^+ : \mu \geq \beth_{(2^{\kappa+|T|})^+}\}$ is empty
or is $\{\mu^+ : \mu \geq \beth_{(2^{\kappa+|T|})^+}\}$ (it relies on some developments from [Sh 300] but is
self-contained).

It was then reasonable to deal with weakening the requirement on κ to measur-
ability. Kolman Shelah [KlSh 362] proved that if $\mu \in \mathrm{Cat}_T$, <u>then</u> (after cosmetic
changes), for the right $\leq_T$ the class $\{M : M \models T, \|M\| < \lambda\}$ has amalgamation
and joint embedding property. This is continued in [Sh 472] which gets results on
categoricity parallel to the one in [MaSh 285] for the "downward" implication.

In [Sh 88] we deal with abstract elementary classes (they include models of
$T \subseteq L_{\kappa, \aleph_0}$, see 0.1), prove a representation theorem (see 0.5 below), and investigate
categoricity in $\aleph_1$ (and having models in $\aleph_2$, limit models, realizing and materializ-
ing types). Unfortunately, we do not have anything interesting to say here on this
context. So we add amalgamation and the joint embedding properties thus getting
to the framework of Jonsson [J] (they are the ones needed to construct homoge-
neous universal models). So this context is more narrow than the ones discussed
above, but we do not use large cardinals. We concentrate here, for categoricity on
λ, on the case "λ is regular". See for later works [Sh 576], [Sh 600] and [ShVi 635].
We quote the basics from [Sh 88] (or [Sh 576]).
We thank Andres Villaveces and Rami Grossberg for much help.

0.1 Definition. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an abstract elementary class <u>if</u> for some vocabulary
$\tau = \tau(K) = \tau(\mathfrak{K}), K$ is a class of $\tau(K)$-models, and the following axioms hold.

$Ax0$: The holding of $M \in K, N \leq_{\mathfrak{K}} M$ depends on N, M only up to isomorphism i.e. $[M \in K, M \cong N \Rightarrow N \in K]$, and [if $N \leq_{\mathfrak{K}} M$ and f is an isomorphism from M onto the τ-model M' mapping N onto N' <u>then</u> $N' \leq_{\mathfrak{K}} M'$].

AxI: If $M \leq_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

$AxII$: $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ implies $M_0 \leq_{\mathfrak{K}} M_2$ and $M \leq_{\mathfrak{K}} M$ for $M \in K$.

$AxIII$: If λ is a regular cardinal, M_i (for $i < \lambda$) is a $\leq_{\mathfrak{K}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{K}} M_j$) and continuous (i.e. for limit ordinal $\delta < \lambda$ we have $M_\delta = \bigcup_{i<\delta} M_i$) <u>then</u> $M_0 \leq_{\mathfrak{K}} \bigcup_{i<\lambda} M_i \in \mathfrak{K}$.

$AxIV$: If λ is a regular cardinal, $M_i (i < \lambda)$ is $\leq_{\mathfrak{K}}$-increasing continuous and $M_i \leq_{\mathfrak{K}} N$ <u>then</u> $\bigcup_{i<\lambda} M_i \leq_{\mathfrak{K}} N$.

AxV: If $M_0 \subseteq M_1$ and $M_\ell \leq_{\mathfrak{K}} N$ for $\ell = 0, 1$, <u>then</u> $M_0 \leq_{\mathfrak{K}} M_1$.

$AxVI$: $LS(\mathfrak{K})$ exists[1]; see below Definition 0.3.

0.2 Definition. 1) $K_\mu =: \{M \in K : \|M\| = \mu\}$.
2) We say h is a $\leq_{\mathfrak{K}}$-embedding of M into N is for some $M' \leq_{\mathfrak{K}} N, h$ is an isomorphism from M onto M'.

0.3 Definition. 1) We say that μ is a Skolem Lowenheim number of $\mathfrak{K}$ <u>if</u> $\mu \geq \aleph_0$ and:

$(*)_K^\mu$ for every $M \in K, A \subseteq M, |A| \leq \mu$ there is $M', A \subseteq M' \leq_{\mathfrak{K}} M$ and $\|M'\| \leq \mu$.

2) $LS'(\mathfrak{K}) = \text{Min}\{\mu : \mu$ is a Skolem Lowenheim number of $\mathfrak{K}\}$.
3) $LS(\mathfrak{K}) = LS'(\mathfrak{K}) + |\tau(K)|$.

0.4 Claim. *1) If I is a directed partial order, $M_t \in K$ for $t \in I$ and $s <_I t \Rightarrow M_s \leq_{\mathfrak{K}} M_t$ <u>then</u>*

   (a) $M_s \leq_{\mathfrak{K}} \bigcup_{t \in I} M_t \in K$ for every $s \in I$

   (b) if $(\forall t \in I)[M_t \leq_{\mathfrak{K}} N]$ then $\bigcup_{t \in I} M_t \leq_{\mathfrak{K}} N$.

2) If $A \subseteq M \in K, |A| + LS'(\mathfrak{K}) \leq \mu \leq \|M\|$, <u>then</u> there is $M_1 \leq_{\mathfrak{K}} M$ such that $\|M_1\| = \mu$ and $A \subseteq M_1$.
3) If I is a directed partial order, $M_t \leq N_t \in K$ for $t \in I$ and $s \leq_I t \Rightarrow M_s \leq_{\mathfrak{K}} M_t$ & $N_s \leq_{\mathfrak{K}} N_t$ <u>then</u> $\bigcup_t M_t \leq_{\mathfrak{K}} \bigcup_t N_t$.

---

[1]We normally assume $M \in \mathfrak{K} \Rightarrow \|M\| \geq LS(\mathfrak{K})$, here there is no loss in it. It is also natural to assume $|\tau(\mathfrak{K})| \leq LS(\mathfrak{K})$ which just means increasing $LS(\mathfrak{K})$.

0.5 Claim. *Let $\mathfrak{K}$ be an abstract elementary class. There are τ^+, Γ such that:*

(a) *τ^+ is a vocabulary extending $\tau(K)$ of cardinality $LS(\mathfrak{K})$*

(b) *Γ is a set of quantifier free types in τ^+ (each is an m-type for some $m < \omega$)*

(c) *$M \in K$ iff for some τ^+-model M^+ omitting every $p \in \Gamma$ we have*
   $M = M^+ \restriction \tau$

(d) *$M \leq_{\mathfrak{K}} N$ iff there are τ^+-models M^+, N^+ omitting every $p \in \Gamma$ such that*
   $M^+ \subseteq N^+, M = M^+ \restriction \tau(K)$ and $N = N^+ \restriction \tau(K)$.
   We can replace $M \leq_{\mathfrak{K}} N$ by $M_i \leq_{\mathfrak{K}} N$ for a family $\{M_i : i \in I\}$ (getting τ^+-expansions M_i^+, N^+ of M_i, N respectively, such that $M_i^+ \subseteq N^+$ and M_i^+, N^+ omit every $p \in \Gamma$ for every $i \in I$) if for any $\bar{a} \in {}^{\omega>}M$ for some $i, \bar{a} \in {}^{\omega>}(M_i)$ and $\bar{a} \in {}^{\omega>}(M_j) \Rightarrow M_i \subseteq M_j$

(e) *if $M \leq_{\mathfrak{K}} N$ and M^+ is an expansion of M to a τ^+-model omitting every $p \in \Gamma$ then we can find a τ^+-expansion of N omitting every $p \in \Gamma$ such that $M^+ \subseteq N^+$.*

0.6 Claim. *Assume $\mathfrak{K}$ has a member of cardinality $\geq \beth_{(2^{LS(\mathfrak{K})})^+}$ (here and else-where we can weaken this to: has a model of cardinality $\geq \beth_\alpha$ for every $\alpha < (2^{LS(\mathfrak{K})})^+$). Then there is Φ proper for linear orders (see [Sh:c, Ch.VII,§2]) such that:*

(a) *$|\tau(\Phi)| = LS(\mathfrak{K})$*

(b) *for linear orders $I \subseteq J$ we have*
   $EM_\tau(I, \Phi) \leq_{\mathfrak{K}} EM(J, \Phi)(\in K)$.

(c) *$EM_\tau(I, \Phi)$ has cardinality $|I| + LS(\mathfrak{K})$ (so $\mathfrak{K}$ has a model in every cardinality $\geq LS(\mathfrak{K})$).*

PART 1
§1 The Framework

1.1 Hypothesis.

    (a) $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ an abstract elementary class (0.1) so
        $K_\lambda = \{M \in K : \|M\| = \lambda\}$

    (b) $\mathfrak{K}$ has amalgamation and the joint embedding property

    (c) K has members of arbitrarily large cardinality, equivalently: K has a member of cardinality at least $\beth_{(2^{LS(\mathfrak{K})})^+}$.

1.2 Convention. 1) So there is a monster $\mathfrak{C}$ (see [Sh:a, Ch.I,§1] = [Sh:c, Ch.I,§1]).

1.3 Definition. We say K (or $\mathfrak{K}$) is categorical in λ <u>if</u> it has one and only one model of cardinality λ, up to isomorphism.

1.4 Definition. 1) We can define $\mathrm{tp}(\bar{a}, M, N)$ (when $M \leq_{\mathfrak{K}} N$ and $\bar{a} \subseteq N$), as $(\bar{a}, M, N)/E$ where E is the following equivalence relation: $(\bar{a}^1, M^1, N^1)\, E\, (\bar{a}^2, M^2, N^2)$ <u>iff</u> $M^\ell \leq_{\mathfrak{K}} N^\ell, \bar{a}^\ell \in {}^\alpha(N^\ell)$ (for some α) and $M^1 = M^2$ and there is $N \in K$ satisfying $M^1 = M^2 \leq_{\mathfrak{K}} N$ and $\leq_{\mathfrak{K}}$-embedding $f^\ell : N^\ell \to N$ over M^ℓ (i.e. $f \upharpoonright M^\ell$ is the identity) for $\ell = 1, 2$ and $f^1(\bar{a}^1) = f^2(\bar{a}^2)$.
2) We omit N when $N = \mathfrak{C}$ (see 1.2) and may then write $\frac{\bar{a}}{M} = \mathrm{tp}(\bar{a}, M, \mathfrak{C})$. We can define N is κ-saturated (when $\kappa > LS(\mathfrak{K})$) by: if $M \leq_{\mathfrak{K}} N, \|M\| < \kappa$ and $p \in \mathscr{S}^{<\omega}(M)$ (see below) <u>then</u> p is realized in M, i.e. for some $\bar{a} \subseteq N, p = \mathrm{tp}(\bar{a}, M, N)$.
3) $\mathscr{S}^\alpha(M) = \{\mathrm{tp}(\bar{a}, M, N) : \bar{a} \in {}^\alpha N, M \leq_{\mathfrak{K}} N\}$; we define $p \upharpoonright M$ when $M \leq_{\mathfrak{K}} N$ & $p \in \mathscr{S}(N)$ as $\mathrm{tp}(\bar{a}, M, N_1)$ when $N \leq_{\mathfrak{K}} N_1, p = \mathrm{tp}(\bar{a}, N, N_1)$. Let $p \leq q$ mean $p \in \mathscr{S}(M), q \in \mathscr{S}(N), p = q \upharpoonright M$; see [Sh 300, Ch.II] or [Sh 576, §0].
4) $\mathscr{S}(M) = \mathscr{S}^1(M)$ (could just as well use $\mathscr{S}^{<\omega}(M) = \bigcup_{n<\omega} \mathscr{S}^n(M)$).
5) If $M_0 \leq_{\mathfrak{K}} M_1$ and $p_\ell \in S^\alpha(M_\ell)$ for $\ell = 1, 2$, <u>then</u> $p_0 = p_1 \upharpoonright M_0$ means that for some $\bar{a}, N$ we have $M_1 \leq_{\mathfrak{K}} N$ and $\bar{a} \in {}^\alpha N$ and $p_\ell = \mathrm{tp}(\bar{a}, M_\ell, N)$ for $\ell = 1, 2$.

1.5 Definition. Let $\mathfrak{K}$ stable in λ mean: $\|M\| \leq \lambda \Rightarrow |\mathscr{S}(M)| \leq \lambda$ and $\lambda \geq LS(\mathfrak{K})$.

1.6 Convention. If not said otherwise, Φ is as in 0.6.

1.7 Claim. *If K is categorical in λ and $\lambda \geq LS(\mathfrak{K})$, <u>then</u>*

    (a) *$\mathfrak{K}$ is stable in every μ which satisfifes $LS(\mathfrak{K}) \leq \mu < \lambda$, hence*

    (b) *the model $M \in K_\lambda$ is cf(λ)-saturated (if cf(λ) > $LS(\mathfrak{K})$).*

Proof. Like [KlSh 362].

1.8 Definition. E_μ is the following relation,

$p\ E_\mu\ q$ iff for some $M \in K, m < \omega$ we have
$$p, q \in \mathscr{S}^m(M) \text{ and } [N \leq_{\mathfrak{K}} M \ \& \ \|N\| \leq \mu \Rightarrow p \restriction N = q \restriction N].$$

Obviously it is an equivalence relation.

1.9 Remark. 1) In previous contexts $E_{LS(\mathfrak{K})}$ is equality, e.g. the axioms of NF in [Sh 300, Ch.II,§1] implies it; but here we do not know — this is the main difficulty. We may look at this as our bad luck, or inversely, a place to encounter some of the difficulty of dealing with $L_{\mu,\omega}$ (in which our context is included).
2) In the cases we shall deal with we can define "$M \in K_{LS(\mathfrak{K})}$" is saturated.

1.10 Claim. *1) There is no maximal member in K, in fact for every $M \in K$ there is $N, M <_{\mathfrak{K}} N \in K, \|N\| \leq \|M\| + LS(\mathfrak{K})$, hence for every $\lambda \geq \|M\| + LS(\mathfrak{K})$ there is $N \in K_\lambda$ such that $M <_{\mathfrak{K}} N \in K_\lambda$.*
2) If $p_2 \in \mathscr{S}^\alpha(M_2)$ and $M_1 \leq_{\mathfrak{K}} M_2 \in K$ <u>then</u> for one and only one $p_1 \in \mathscr{S}^\alpha(M_1)$ we have $p_1 = p_2 \restriction M_1$.
3) If $p_1 \in \mathscr{S}^\alpha(M_1)$ and $M_1 \leq_{\mathfrak{K}} M_2 \in K$ <u>then</u> for some $p_2 \in \mathscr{S}^\alpha(M_2)$ we have $p_1 = p_2 \restriction M_1$.
4) If $M_1 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$ and $p_\ell \in \mathscr{S}^\alpha(M_\ell)$ for $\ell = 1, 2, 3$ <u>then</u> $p_3 \restriction M_2 = p_2$ \& $p_2 \restriction M_1 = p_1 \Rightarrow p_3 \restriction M_1 = p_2$.

Proof. 1) Immediate by clause (c) of the hypothesis 1.1 and claim 0.6.
2) Straightforward.
3) By amalgamation.
4) Check. $\square_{1.10}$

1.11 Claim. *If $\langle M_i : i \leq \omega \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and $p_n \in \mathscr{S}^\alpha(M_n)$ and $p_n = p_{n+1} \restriction M_n$ for $n < \omega$, <u>then</u> there is $p_\omega \in \mathscr{S}^\alpha(M_\omega)$ such that $n < \omega \Rightarrow p_\omega \restriction_n = p_n$.*

Proof. Straight chasing diagrams.

1.12 Remark. In 1.11 we do not claim uniqueness and not existence replacing ω for δ of uncountable cofinality. In general not true [Saharon add].

§2 Variant of Saturated

2.1 Definition. Assuming $\mathfrak{K}$ stable in μ and α is an ordinal $< \mu^+$, $\mu^+ \times \alpha$ means ordinal product.

1) $M <^\circ_{\mu,\alpha} N$ if: $M \in K_\mu, N \in K_\mu, M \leq_\mathfrak{K} N$ and there is a $\leq_\mathfrak{K}$-increasing sequence $\bar{M} = \langle M_i : i \leq \mu \times \alpha \rangle$ which is continuous, $M_0 = M, M_{\mu \times \alpha} \leq_\mathfrak{K} N$ and every $p \in \mathscr{S}^1(M_i)$ is realized in M_{i+1}.

2) We say $M <^1_{\mu,\alpha} N$ iff $M \in K_\mu, N \in K_\mu, M \leq_\mathfrak{K} N$ and there is a $\leq_\mathfrak{K}$-increasing sequence $\bar{M} = \langle M_i : i \leq \mu \times \alpha \rangle, M_0 = M, M_{\mu \times \alpha} = N$ and every $p \in \mathscr{S}^1(M_i)$ is realized in M_{i+1}.

3) If $\alpha = 1$, we may omit it.

2.2 Lemma. *Assume $\mathfrak{K}$ stable in μ and $\alpha < \mu^+$.*

0) If $\ell \in \{0,1\}$ and $\alpha_1 < \alpha_2 < \mu^+$ and there is $b \subseteq \alpha_2$ such that $otp(b) = \alpha_1$ and $[\ell = 1 \Rightarrow b$ unbounded in $\alpha_2]$ <u>then</u> $<^\ell_{\mu,\alpha_2} \subseteq <^\ell_{\mu,\alpha_1}$.

1) If $M \in K_\mu$, <u>then</u> for some N we have $M <^\circ_{\mu,\alpha} N$ and for some $N, M <^1_{\mu,\alpha} N$.

2)   (a) If $M \in K_\mu, M \leq_\mathfrak{K} M' \leq^\ell_{\mu,\alpha} N$ <u>then</u> $M \leq^\ell_{\mu,\alpha} N$.

*   (b) If $M \in K_\mu, M \leq_\mathfrak{K} M' \leq^\circ_{\mu,\alpha} N' \leq_\mathfrak{K} N \in K_\mu$ <u>then</u> $M \leq^\circ_{\mu,\alpha} N$.*

3) If $\langle M_i : i < \alpha \rangle$ is $\leq_\mathfrak{K}$-increasing sequence in $K_\mu, M_i \leq^\circ_\mu M_{i+1}$ and $\alpha < \mu^+$ is a limit ordinal, <u>then</u> $M_0 \leq^1_{\mu,\alpha} \bigcup_{i<\alpha} M_i$.

4) If $M \leq^\circ_\mu N$ <u>then</u>:

*   (a) any $M' \in K_\mu$ can be $\leq_\mathfrak{K}$-embedded into N (here we can waive $\|M\| = \mu$)*

*   (b) If $M' \leq_\mathfrak{K} N' \in K_{\leq\mu}$, h is a $\leq_\mathfrak{K}$-embedding of M' into M then h can be extended to a $\leq_\mathfrak{K}$-embedding of N' into N.*

5) If $M^\ell \leq^1_{\mu,\kappa} N^\ell$ for $\ell = 1, 2$, h an isomorphism from M^1 into [onto] M^2 then h can be extended to an isomorphism from N^1 into [onto] N^2.

6) If $M \leq^1_{\mu,\kappa} N^\ell$ for $\ell = 1, 2$ <u>then</u> $N^1 \cong N^2$ (even over M).

7) If $M \leq^\circ_{\mu,\kappa} N$, $M \leq_\mathfrak{K} M' \in K_\mu$ <u>then</u> M' can be $<_\mathfrak{K}$-embedded into N over M.

8) If $\mu \geq \kappa > LS(\mathfrak{K})$ and $M <^1_{\mu,\kappa} N$ <u>then</u> N is $cf(\kappa)$-saturated.

Proof. See [Sh 300, Ch.II,3.10,p.319] and around, we shall explain and prove part (8) below.

<u>2.3 Discussion</u>: There (in [Sh 300, Ch.II,3.6]) the main point was that for $\kappa > LS(\mathfrak{K})$, the notions "$\kappa$-homogeneous universal" and κ-saturation (i.e. every "small" 1-type is realized) are equivalent.

Not hard, still [Sh 300, Ch.II,3.6] was a surprise to some. In first order the equivalence saturated $\equiv$ homogeneous universal for $\prec$ seemed a posteriori natural as the homogeneity used was anyhow for sequences of elements realizing the same first order formulas so (forgetting about the models) to some extent this seemed natural; i.e. asking this for any type of 1-element was very natural.

But here, types of 1-element are really meaningful only over a model. So it seems that if over any small submodel every type of 1-element is realized (say in $\mathfrak{A}$) and

we want to embed $N \geq_{\mathfrak{K}} N_0, N_0 \leq_{\mathfrak{K}} \mathfrak{A}$ into $\mathfrak{A}$ over N_0, we encounter the following problem: we cannot continue this as after ω stages, as we get a set which is not a model (if $LS(\mathfrak{K}) > \aleph_0$ this absolutely necessarily fails; and if $LS(\mathfrak{K}) = \aleph_0$ at best the situation is as in [Sh 87a]).

This explains a natural preconception making you not believe; i.e. psychological barrier to prove. It does not mean that the proof is hard.

2.4 Remark. Note that $\leq^1_{\mu,\kappa}, \kappa$ regular are the interesting ones. Still $\leq^0_{\mu,\kappa}$ is enough for universality (2.2(4)) and is natural, $\leq^1_{\mu,\kappa}$ is natural for uniqueness. BUT $<^1_{\mu,\aleph_0} = <^1_{\mu,\aleph_1}$ can be proved only under categoricity (or something like superstability assumptions). LOOK at first order T stable in μ. Then, $M <^1_{\mu,\kappa} N$ is equivalent to:

$$\|M\| = \|N\| = \mu, M, N \models T$$

and there is $\langle M_i : i \leq \kappa \rangle$ which is $\prec$-increasing continuous such that

$$M_0 = M \qquad M_\kappa = N$$

$$(M_{i+1}, c)_{c \in M_i} \text{ is saturated.}$$

Question: Now, is N saturated when $M <^1_{\mu,\kappa} N$?

Answer: It is iff $\mathrm{cf}(\kappa) \geq \kappa_r(T)$. See [Sh:c, Ch.III,§3].

See on limit and superlimit models in [Sh 88].

Before we prove 2.2(8), recall

2.5 Definition. $M \in K$ is κ-saturated if $\kappa > LS(\mathfrak{K})$ and:
$N \leq_{\mathfrak{K}} M, \|N\| < \kappa, p \in \mathscr{S}^1(N) \Rightarrow p$ realized in M.

Proof of 2.2(8).
Statement: If $M <^1_{\mu,\kappa} N$ (κ regular) then N is κ-saturated.

Note: if $\kappa \leq LS(\mathfrak{K})$ the conclusion is essentially empty, but there is no need for the assumption "$\kappa > LS(\mathfrak{K})$".

Proof. Let $\bar{M} = \langle M_i : i \leq \mu \times \kappa \rangle$ witness $M \leq^1_{\mu,\kappa} N$ so $M_0 = M, M_{\mu \times \kappa} = N, M_i \leq_{\mathfrak{K}}$-increasing continuous and every $p \in \mathscr{S}(M_i)$ is realized in M_{i+1}.
Assume

(∗) $N' \leq_{\mathfrak{K}} N, \|N'\| < \kappa, p \in \mathscr{S}(N')$.

We should prove that "p is realized in N". But $\langle M_i : i \leq \mu \times \kappa \rangle$ is increasing continuous

$$\mathrm{cf}(\mu \times \kappa) = \kappa > \|N'\|$$

so $N' \leq_{\mathfrak{K}} M_{\mu \times \kappa} = \bigcup_{i < \mu \times \kappa} M_i$ implies there is $i(*) < \mu \times \kappa$, such that $N' \subseteq M_{i(*)}$
hence by Axiom V we have $N' \leq_{\mathfrak{K}} M_{i(*)}$. So p has (by amalgamation!) an extension
$p^* \in \mathscr{S}(M_{i(*)})$ and p^* is realized in $M_{i(*)+1}$ so in $M_{\mu \times \kappa} = N$. $\square_{2.2}$

Comment: Hence length μ (instead of $\mu \times \kappa$) suffices.
    But for the uniqueness it does not. See 2.2(4) + (5).

Comment: The definition of $\leq^0_{\mu,\kappa}, \leq^1_{\mu,\kappa}$ is also essentially taken from
[Sh 300, Ch.II,3.10]. We need the intermediate steps to construct models so we
have to have μ of them in order to deal with all the elements.

2.6 Claim. *If K is categorical in $\lambda, M \in K_\lambda$ and $cf(\lambda) > \mu$ <u>then</u>:*
if $N <_{\mathfrak{K}} M \in K_\lambda, N \in K_\mu, N' <_{\mathfrak{K}} M, h$ an isomorphism from N onto N', <u>then</u> h
can be extended to an automorphism of M.

Proof. By 1.4 we have $LS(\mathfrak{K}) \leq \mu < \lambda \Rightarrow \mathfrak{K}$ stable in λ. We can find $\langle M_i : i < \lambda \rangle$
which is $<_{\mathfrak{K}}$-increasing continuous, $\|M_i\| = |i| + LS(\mathfrak{K})$,
$M_i <^1_{|i|+LS(\mathfrak{K}),|i|+LS(\mathfrak{K})} M_{i+1}$. By the categoricity assumption without loss of gen-
erality $M = \bigcup_{i<\lambda} M_i$. As $cf(\lambda) > \mu$ for some $i_0 < \lambda$ we have $N, N' \prec M_{i_0}$.
    By 2.2 we can build an automorphism. $\square_{2.6}$

2.7 Definition. For $\mu \geq LS(\mathfrak{K})$, we say $N \in K_\mu$ is (μ, κ)-saturated <u>if</u> for some M
we have $M <^1_{\mu,\kappa} N$ (so κ is $\leq \mu$, normally regular).

2.8 Claim. *1) The (μ, κ)-saturated model is unique (even over M) if it exists at*
all.
2) If M is (μ, κ)-saturated, $\kappa = cf(\kappa)$, $cf(\kappa) > LS(\mathfrak{K})$ <u>then</u> M is κ-saturated.
3) If M is (μ, κ)-saturated for every $\kappa = cf(\kappa) \leq \mu$ and $\mu > LS(\mathfrak{K})$ <u>then</u> M is
μ-saturated.

Discussion: It is natural to define saturated as $\|M\|$-saturated. (I may have confu-
sions using the other being (μ, κ)-saturated for every regular $\kappa \leq \mu$.) This is partic-
ularly reasonable when the cardinal is regular, e.g. if K categorical in $\lambda, \lambda = cf(\lambda)$
the model in K_λ is λ-saturated.
    Part of the program is to prove that all the definitions are equivalent.
    For now in Definition 2.7 we are not sure that such a model exists.

§3 Splitting

Whereas non-forking is very nice in [Sh:c], in more general contexts, non first order, it is not clear whether we have so good a notion, hence we go back to earlier notions from [Sh 3], like splitting. It still gives for many cases $p \in \mathscr{S}(M)$, a "definition" of p over some "small" $N \leq_{\mathfrak{K}} M$. We need μ-splitting because $E_{LS(\mathfrak{K})}$ is not known to be equality (see 1.8).

3.1 Context. Inside the monster model $\mathfrak{C}$.

3.2 Definition. $p \in \mathscr{S}(M)$ does μ-split over $N \leq_{\mathfrak{K}} M$ if:

> $\|N\| \leq \mu$, and there are N_1, N_2, h such that:
> $\|N_1\| = \|N_2\| \leq \mu$ and $N \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} M$, for $\ell = 1, 2$
> h an elementary mapping from N_1 onto N_2 over N such that
> the types $p \restriction N_2$ and $h(p \restriction N_1)$ are contradictory and $N \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} M$.

3.3 Claim. *1) Assume $\mathfrak{K}$ is stable in $\mu, \mu \geq LS(\mathfrak{K})$. If $M \in \mathfrak{K}_{\geq \mu}$ and $p \in \mathscr{S}^1(M)$, then for some $N_0 \subseteq M, \|N_0\| = \mu, p$ does not μ-split over N_0 (see Definition 3.2).*
2) Moreover, if $2^\kappa > \mu, \langle M_i : i \leq \kappa + 1 \rangle$ is $<_{\mathfrak{K}}$-increasing, $\bar{a} \in {}^m(M_{\kappa+1})$, $tp(\bar{a}, M_{i+1}, M_{\kappa+1})$ does $(\leq \mu)$-split over M_i, then $\mathfrak{K}$ is not stable in μ.

Proof of 3.3. 1) If not, we can choose by induction on $i < \mu$ N_i, N_i^1, N_i^2, h_i such that:

(a) $\langle N_i : i \leq \mu \rangle$ is increasing continuous, $N_i <_{\mathfrak{K}} M, \|N_i\| = \mu$

(b) $N_i \leq_{\mathfrak{K}} N_i^\ell \leq_{\mathfrak{K}} N_{i+1}$

(c) h_i is an elementary mapping from N_i^1 onto N_i^2 over N_i,

(d) $p \restriction N_i^2, h_i(p \restriction N_i^1)$ are contradictory, equivalently distinct (we could have defined them for $i < \mu^+$).

Let $\chi = \mathrm{Min}\{\chi : 2^\chi > \mu\}$ so $2^{<\chi} \leq \mu$. Now contradict stability in μ as in part (2).
2) Similar to [Sh:a, Ch.I,§2] or [Sh:c, Ch.I,§2] (by using models), but we give details. Without loss of generality $M_i \in K_{\leq \mu}$ for $i \leq \kappa + 1$. For each $i < \kappa$ let $N_{i,1}, N_{i,2}$ be such that $M_i \leq_{\mathfrak{K}} N_{i,\ell} \leq_{\mathfrak{K}} M_{i+1}, g_i$ an isomorphism from $N_{i,1}$ onto $N_{i,2}$ over M_i and $tp(\bar{a}, N_{i,2}) \neq g_i(tp(\bar{a}, N_{i,1}))$. Without loss of generality $2^{<\kappa} \leq \mu$. We define by induction on $\alpha \leq \kappa$ a model M_α^* and for each $\eta \in {}^\alpha 2$, a mapping h_η such that:

(a) $M_\alpha^* \in K_\mu$ is $\leq_{\mathfrak{K}}$-increasing continuous

(b) for $\eta \in {}^\alpha 2, h_\eta$ is a $\leq_{\mathfrak{K}}$-embedding of M_α into M_α^*

(c) if $\beta < \alpha, \eta \in {}^\alpha 2$, then $h_{\eta \restriction \beta} \subseteq h_\eta$

(d) if $\alpha = \beta + 1, \nu \in {}^\beta 2$, then $h_{\nu \hat{} <0>}(N_{i,1}) = h_{\nu \hat{} <1>}(N_{i,2})$.

There is no problem to carry the definition (we are using amalgamation only in $K_{\leq \mu}$ and if we start with $M_0 \in K_\mu$ only in K_μ). Now for each $\eta \in {}^\kappa 2$ we can find $M_\eta^* \in K_\mu, M_\kappa^* \leq_{\mathfrak{K}} M_\eta^*$ and $\leq_{\mathfrak{K}}$-embedding h_η^+ of $M_{\kappa+1}$ into M_η^* extending

$h_\eta = \bigcup_{\alpha<\kappa} h_{\eta\restriction\alpha}$. Now $\{\mathrm{tp}(h_\eta^+(\bar{a}), M_\kappa^*, M_\eta^*) : \eta \in {}^\kappa 2\}$ is a family of $2^\kappa > \mu$ distinct members of $\mathscr{S}^m(M_\kappa^*)$ and recall $M_\kappa^* \in K_\mu$ so we are done. $\qquad\qquad \square_{3.3}$

3.4 Conclusion. If $p \in \mathscr{S}^m(M)$, M is μ^+-saturated, $\kappa = \mathrm{cf}(\kappa) \leq \mu$, <u>then</u> for some $N_0 <_{\mu,\kappa}^\circ N_1 \leq_\mathfrak{K} M$, (so $\|N_1\| = \mu$) we have:

p is the E_μ-unique extension of $p \restriction N_1$ which does not μ-split over N_0.

§4 Indiscernibles and E.M. Models

4.1 Definition. Let $h_i : Y \to \mathfrak{C}$ for $i < i^*$.
1) $\langle h_i : i < i^* \rangle$ is an indiscernible sequence (of character $< \kappa$) (over A) if <u>for every</u> g, a partial one to one order preserving map from i^* to i^* (of cardinality $< \kappa$) <u>there is</u> $f \in AUT(\mathfrak{C})$, such that

$$g(i) = j \Rightarrow h_j \circ h_i^{-1} \subseteq f$$

$$(\text{and } \mathrm{id}_A \subseteq f).$$

2) $\langle h_i : i < i^* \rangle$ is an indiscernible set (of character κ) (over A) if: for every g partial one to one map from i^* to i^* (with $|\mathrm{Dom}\, g| \leq \kappa$) <u>there is</u> $f \in AUT(\mathfrak{C})$, such that

$$g(i) = j \Rightarrow h_j \circ h_i^{-1} \subseteq f$$

$$(\text{and } \mathrm{id}_A \subseteq f).$$

3) $\langle h_i : i < i^* \rangle$ is a strictly indiscernible sequence, if $i^* \geq \omega$ and for some Φ, proper for linear orders (see [Sh:a, Ch.VII] or [Sh:c, Ch.VII]) in vocabulary $\tau_1 = \tau(\Phi)$ extending $\tau(K)$, there is $M^1 = EM^1(i^*, \Phi)$ such that M^1 is the Skolem Hull of $\{x_i : i < i^*\}$, and a sequence of unary terms $\langle \sigma_t : t \in Y \rangle$ such that:

$$\sigma_t(x_i) = h_i(t) \text{ for } i < i^*, t \in Y$$

$$M^1 \restriction \tau(K) <_{\mathfrak{K}} \mathfrak{C}.$$

4) Let $h_i : Y_i \to \mathfrak{C}$ for $i < i^*$ we say that $\langle h_i : i < i^* \rangle$ has characteristic σ if:

(∗) if $h_i' : Y_i \to \mathfrak{C}$ for $i < i^*$ and for every $u \in [i^*]^{<\sigma}$ there is an automorphism f_u of $\mathfrak{C}$ such that $f_u \restriction A = \mathrm{id}_A$ and $i \in u \Rightarrow f_u \circ h_i = h_i'$, <u>then</u> there is an automorphism f of $\mathfrak{C}$ such that $f \restriction A = \mathrm{id}_A$ and $i < i^* \to f \circ h_i = h_i'$.

4.2 Notation. We can replace h_i by the sequence $\langle h_i(t) : t \in Y \rangle$.

4.3 Definition. 1) $\mathfrak{K}$ has the (κ, θ)-order property <u>if</u> for every α there are $A \subseteq \mathfrak{C}$ and $\langle \bar{a}_i : i < \alpha \rangle$, where $\bar{a}_i \in {}^{\kappa}\mathfrak{C}$ and $|A| \leq \theta$ such that:

(∗) if $i_0 < j_0 < \alpha, i_1 < j_1 < \alpha$ then for no $f \in AUT(\mathfrak{C})$ do we have
$f \restriction A = \mathrm{id}_A, f(\bar{a}_{i_0} \hat{\,} \bar{a}_{j_0}) = \bar{a}_{j_1} \hat{\,} \bar{a}_{i_1}$.

If $A = \emptyset$ i.e. $\theta = 0$, we write "κ-order property".
2) $\mathfrak{K}$ has the $(\kappa_1, \kappa_2, \theta)$ order property <u>if</u> for every α there are $A \subseteq \mathfrak{C}$ satisfying $|A| \leq \theta, \langle \bar{a}_i : i < \alpha \rangle$ where $\bar{a}_i \in {}^{\kappa_1}\mathfrak{C}$ and $\langle \bar{b}_i : i < \alpha \rangle$ where $\bar{b}_i \in {}^{\kappa_2}\mathfrak{C}$ such that

(∗) if $i_0 < j_0 < \alpha, i_1 < j_1 < \alpha$, <u>then</u> for no $f \in \mathrm{AUT}(\mathfrak{C})$ do we have
$f \restriction A = \mathrm{id}_A, f(\bar{a}_{i_0}) = \bar{a}_{j_1}, f(\bar{b}_{j_0}) = \bar{b}_{i_1}$.

4.4 Observation. So we have obvious monotonicity properties and if $\theta \leq \kappa$ we can let $A = \emptyset$; so the (κ, θ)-order property implies the $(\kappa + \theta)$-order property.

4.5 Claim. *1) Any strictly indiscernible sequence (over A) is an indiscernible sequence (over A).*
2) Any indiscernible set (over A) is an indiscernible set (over A).

4.6 Claim. *1) If $\mu \geq LS(\mathfrak{K}) + |Y|$ and $h_i^\theta : Y \to \mathfrak{C}$, for $i < \theta < \beth_{(2^\mu)^+}$ (e.g. $h_i^\theta = h_i$) then we can find $\langle h_j' : j < i^* \rangle$, a strictly indiscernible sequence, with $h_j' : Y \to \mathfrak{C}$ such that:*

> (*) *for every $n < \omega, j_1 < \cdots < j_n < i^*$ for arbitrarily large $\theta < \beth_{(2^\mu)^+}$ we can find $i_1 < \cdots < i_n < \theta$ and $f \in AUT(\mathfrak{C})$ such that $h_{j_\ell}' \circ (h_{i_\ell}^\theta)^{-1} \subseteq f$.*

2) If in part (1) for each θ, the sequence $\langle h_j^\theta : j < \theta \rangle$ is an indiscernible sequence of character $\aleph_0$, in (*) any $i_1 < \cdots < i_n < i^*$ will do.
3) In Definition 4.3 we can restrict α to $\alpha < \beth_{(2^{\kappa+\theta+LS(\mathfrak{K})})^+}$ and get an equivalent version.
4) In Definition 4.3 we can demand $\langle \bar{a}\hat{\ }\bar{a}_i : i < \alpha \rangle$ is strictly indiscernible (where $\bar{a}$ lists A) and get an equivalent version.
5) If $\mu \geq LS(\mathfrak{K}) + |Y|, N \leq_{\mathfrak{K}} \mathfrak{C}$ and $h_i^\theta : Y \to N$ for $i < \theta < \beth_{(2^\mu)^+}$ and N^1 is an expansion of N with $|\tau(N^1)| \leq \mu$, then for some expansion N^2 of N^1 with $|\tau(N^2)| \leq \mu$ and Ψ we have:

> (a) *$\tau(\Psi) = \tau(N^2)$*
> (b) *for linear orders $I \subseteq J$ we have*
>   *$EM_{\tau(\mathfrak{K})}(I, \Psi) \leq_{\mathfrak{K}} EM_{\tau(\mathfrak{K})}(I, \Psi) \in K$*
>   *and the skeleton of $EM_{\tau(\mathfrak{K})}(I, \Psi)$ is $\langle \bar{a}_t : t \in I \rangle, \bar{a}_t = \langle a_{t,y} : y \in Y \rangle$*
> (c) *for every $n < \omega$ for arbitrarily large $\theta < \beth_{(2^\mu)^+}$ for some $i_0 < \ldots i_{n-1} < \theta$, for every linear order I and $t_0 < \cdots < t_{n-1}$ in I, letting $J = \{t_0, \ldots, t_{n-1}\}$ there is an isomorphism g from $EM(J, \Psi) \subseteq EM(I, \Psi)$ (those are $\tau(N^2)$-models) onto the submodel of N^2 generated by $\bigcup_{\ell < n} Rang(h_{i_\ell}^\theta)$ such that*
>   *$h_{i_\ell}^\theta(y) = g(a_{t,y})$.*

Proof. As in [Sh:c, Ch.VII,§5] and [Sh 88] [Saharon read], see 8.6 for a similar somewhat more complicated proof.

4.7 Lemma. *1) If there is a strictly indiscernible sequence which is not an indiscernible set of character $\aleph_0$ called $\langle \bar{a}^i : i < \omega \rangle$, then $\mathfrak{K}$ has the $|\ell g(a^i)|$-order property.*
2) If there is $\langle \bar{a}^i : i < i^* \rangle$ is a strictly indiscernible sequence over A of character θ^+ but is not an indiscernible set over A of character θ^+ and $i^* \geq \theta^+$, then $\mathfrak{K}$ has the $(\ell g(\bar{a}^0), |A| + \theta \times \ell g(\bar{a}^0)$-order property.

Remark. Permutation of infinite sets is a more complicated issue.

4.8 Claim. *1) If $\mathfrak{K}$ has the κ-order property <u>then</u>:*

$$I(\chi, \kappa) = 2^\chi \text{ for every } \chi > (\kappa + LS(\mathfrak{K}))^+$$

(and other strong non-structure properties).
2) If $\mathfrak{K}$ has the $(\kappa_1, \kappa_2, \theta)$-order property and $\chi \geq \kappa = \kappa_1 + \kappa_2 + \theta$ <u>then</u> for some $M \in K_\chi$, we have $|\mathscr{S}^{\kappa_2}(M)/E_\kappa| > \chi$.

Proof. 1) By [Sh:e, Ch.III,§3] (preliminary version appears in [Sh 300, Ch.III,§3]) (note the version on e.g. $\triangle(L_{\lambda^+,\omega})$).
2) Straight. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square_{4.8}$

4.9 Definition. 1) Suppose $M \leq_{\mathfrak{K}} N$ and $p \in \mathscr{S}^m(N)$. Then p divides over M if there are elementary maps $\langle h_i : i < \bar{\kappa} \rangle$, $\mathrm{Dom}(h_i) = N$, $h_i \restriction M = \mathrm{id}_M$, $\langle h_i : i < \bar{\kappa} \rangle$ is a strictly indiscernible sequence and $\{h_i(p) : i < \bar{\kappa}\}$ is contradictory i.e. no element (in some $\mathfrak{C}', \mathfrak{C} <_{\mathfrak{K}} \mathfrak{C}'$) realizing all of them; recall $\bar{\kappa}$ is the cardinality of $\mathfrak{C}$. Let μ-divides mean no elements realize $\geq \mu$ of them.
2) $\kappa_\mu(\mathfrak{K})$ [or $\kappa_\mu^*(\mathfrak{K})$] is the set of regular κ such that for some $\leq_{\mathfrak{K}}$-increasing continuous $\langle M_i : i \leq \kappa+1 \rangle$ in K_μ and $b \in M_{\kappa+1}$ for every $i < \kappa$ we have: $\mathrm{tp}(b, M_\kappa, M_{\kappa+1})$ [or $\mathrm{tp}(b, M_{i+1}, M_{\kappa+1})$] divides over M_i; so $\kappa \leq \mu$.
3) $\kappa_{\mu,\theta}(\mathfrak{K})$ [or $\kappa_{\mu,\theta}^*(\mathfrak{K})$] is the set of regular κ such that for some $\leq_{\mathfrak{K}}$-increasing continuous sequence $\langle M_i : i \leq \kappa+1 \rangle$ in K_θ and $b \in M_{\kappa+1}$ for every $i < \kappa$ we have: $\mathrm{tp}(b, M_\kappa, M_{\kappa+1})$ [or $\mathrm{tp}(b, M_{i+1}, M_{\kappa+1})$], μ-divides over M_i, so $\kappa \leq \theta$ (see Definition 4.12 below).

4.10 Remark. 1) A natural question: is there a parallel to forking?
2) Note the difference between $\kappa_\mu(\mathfrak{K})$ and $\kappa_\mu^*(\mathfrak{K})$. Note that now the "local character" is apparently lost.

4.11 Fact. 1) In Definition 4.9(1) we can equivalently demand: no element realizing $\geq \beth_{(2^\chi)^+}$ of them, where $\chi = \|N\|$.
2) If $\kappa \in \kappa_\mu^*(\mathfrak{K}), \theta = \mathrm{cf}(\theta) \leq \kappa$ <u>then</u> $\theta \in \kappa_\mu^*(\mathfrak{K})$ and similarly of $\kappa_{\mu,\theta}^*(\mathfrak{K})$.
3) $\kappa_\mu^*(\mathfrak{K}) \subseteq \kappa_\mu(\mathfrak{K})$ similarly $\kappa_{\mu,\theta}^*(\mathfrak{K}) \subseteq \kappa_{\mu,\theta}(\mathfrak{K})$.

4.12 Definition. Suppose $M \leq_{\mathfrak{K}} N, p \in \mathscr{S}(N), M \in K_{\leq\mu}, \mu \geq LS(\mathfrak{K})$.
1) We say p does μ-strongly splits over M, <u>if</u> there are $\langle \bar{a}^i : i < \omega \rangle$ such that:
   (i) $\bar{a}^i \in {}^{\gamma\geq}\mathfrak{C}$ for $i < \omega$, $\gamma < \mu^+$, $\langle \bar{a}^i : i < \omega \rangle$ is strictly indiscernible over M
   (ii) for no b realizing p do we have $\mathrm{tp}(\bar{a}^0 {}^\frown \langle b \rangle, M, \mathfrak{C}) = \mathrm{tp}(\bar{a}^1 {}^\frown \langle b \rangle, M, \mathfrak{C})$.

2) We say p explicitly μ-strongly splits over M if in addition $\bar{a}^0 \cup \bar{a}^1 \subseteq N$.
3) Omitting μ means any μ (equivalently $\mu = \|N\|$).

4.13 Claim. *1) Strongly splitting implies dividing with models of cardinality $\leq \mu$ if $(*)_\mu$ holds where $(*)_\mu = (*)_{\mu,\aleph_0,\aleph_0}$ and*

$(*)_{\mu,\theta,\sigma}$ *If $\langle \bar{a}^i : i < i^* \rangle$ is a strictly indiscernible sequence, $\bar{a}^i \in {}^\mu\mathfrak{C}, \bar{b} \in {}^{\sigma>}\mathfrak{C}$, <u>then</u> for some $u \subseteq i^*$, $|u| < \theta$ and the isomorphism type of $(\mathfrak{C}, \bar{a}^i {}^\frown \bar{b})$ for all $i \in i^* \backslash u$ is the same.*

4.14 Claim. *1) Let $\mu(*) = \mu + \sigma + LS(\mathfrak{K})$. Assume $\langle \bar{a}^i : i < i^* \rangle$ and $\bar{b}$ form a counterexample to $(*)_{\mu,\theta,\sigma}$ of 4.13 and $\theta \geq \beth_{(2^{\mu(*)})^+}$ <u>then</u> $\mathfrak{K}$ has the $\mu(*)$-order property.*
2) We can also conclude that for $\chi \geq \mu + LS(\mathfrak{K})$, for some $M \in K_\chi$ we have $|\mathscr{S}^{\ell g(\bar{b})}(M)| > \chi$.
3) If we have "$\theta < \beth_{(2^{\mu(*)})^+}$" we can still get that for every $\chi \geq \mu + \sigma + LS(\mathfrak{K}) + \theta$ for some $M \in K_\chi$, we have $|\mathscr{S}^{\ell g(\bar{b})}(M)| \geq \chi^\theta$.
4) In part (1) it suffices to have such an example for every $\theta < \beth_{(2^{\mu(*)})^+}$, of course, for fixed $\mu(*)$.

Proof. Straight, using 4.15 below.

4.15 Claim. *Assume $M = EM(I, \Phi), LS(\mathfrak{K}) + \ell g(\bar{a}_i) \leq \mu, \mu \geq |\alpha| + LS(\mathfrak{K})$ and $M \leq_\mathfrak{K} N, \bar{b} \in {}^\alpha N$ and*

   (∗) for no $J \subseteq I, |J| < \beth_{(2^\mu)^+}$ do we have for all $t, s \in I \backslash J$,
   $tp(\bar{a}_t {}^\frown \bar{b}, \emptyset, N) = tp(\bar{a}_s {}^\frown \bar{b}, \emptyset, N).$

<u>Then</u>

   (A) *we can find Φ' proper for linear orders and a formula φ (not necessarily first order, but $\pm\varphi$ is preserved by $\leq_\mathfrak{K}$-embeddings) such that for any linear order I'*
   $M = EM(I', \Phi'), \bar{a}_t = \bar{a}^t {}^\frown \bar{b}_t, \ell g(\bar{a}^t) \leq \mu, \ell g(\bar{b}_t) = \alpha$ *and*
   $M \models \varphi[\bar{a}^t, \bar{b}_s] \Leftrightarrow t < s$
   (if $\alpha < \omega$, this is half the finitary order property)

   (B) *this implies instability in every $\mu' \geq \mu$ if $\alpha < \omega$*

   (C) *this implies the $(\mu + |\alpha|)$-order property and even the $(\mu, |\alpha|, 0)$-order property*

   (D) *if $\bar{b} \in {}^\alpha M$ then "$|J| < \mu^+$" or just "$|J| < |\alpha|^+ + \aleph_0$" in (∗) suffices*

   (E) *if $\chi \geq \mu$, for some $M \in K_\chi$, <u>then</u> $|\mathscr{S}^\alpha(M)| > \chi$ moreover $|\mathscr{S}^\alpha(M)/E_\mu| > \chi$.*

Proof. As we can increase I, without loss of generality the linear order I is dense with no first or last element and is $(\beth_{(2^\mu)^+})^+$-strongly saturated, see Definition 4.17 below. So for some p and some interval I_0 of I, the set $Y_0 = \{t \in I_0 : tp(\bar{a}_t {}^\frown \bar{b}, \emptyset, N) = p\}$ is a dense subset of I_0. Also for some $q \in \mathscr{S}^\alpha(M) \backslash \{p\}$, the set $Y_1 = \{t \in I : tp(\bar{a}_t {}^\frown \bar{b}, \emptyset, N) = q\}$ has cardinality $\geq \beth_{(2^\mu)^+}$ and let $Y_1' \subseteq Y_1$ have cardinality $\beth_{(2^\mu)^+}$. As we can shrink I_0 without loss of generality I_0 is disjoint from Y_1' and as we can shrink Y_1 without loss of generality $(\forall s \in Y_1')(\forall t \in I_0)(s <^I t)$ or $(\forall s \in Y_q')(\forall t \in I_0)(t <^I s)$.
By the Erdös-Rado theorem, for every $\theta < \beth_{(2^\mu)^+}$ there are $s_\alpha^\theta \in Y_1'$ for $\alpha < \theta$ such that $\langle s_\alpha^\theta : \alpha < \theta \rangle$ is strictly increasing or strictly decreasing; without loss of generality the case does not depend on θ, so as we can invert I without loss of generality it is increasing. Let $t_\alpha^* \in Y_1'$ for $\alpha < \beth_{(2^\mu)^+}$ be strictly increasing. Hence (try $(p_1, p_2) = (p, q)$ and $(p_1, p_2) = (q, p)$, one will work)

(∗) we can find $p_1 \neq p_2$ such that

(∗∗) for every $\theta < \beth_{(2^\mu)^+}$ there is an increasing sequence $\langle t_\alpha^\theta : \alpha < \theta + \theta \rangle$ of members of I such that

$$\text{(i)} \qquad \alpha < \theta = \ \text{tp}(\bar{a}_{t_\alpha^\theta} {}^\frown \bar{b}, \emptyset, N) = p_0$$

$$\text{(ii)} \qquad \theta \leq \alpha < \theta + \theta \Rightarrow \ \text{tp}(\bar{a}_{t_\alpha^\theta} {}^\frown \bar{b}, \emptyset, N) = p_1.$$

[Note that we could have replaced "increasing" by

$$\text{(iii)} \qquad \alpha < \beta < \theta \Rightarrow t_\alpha^\theta <_I t_\beta^\theta <_I t_{\theta+\alpha}^\theta <_I t_{\theta+\beta}^\theta.$$

Why? Let $I_1 = \{t \in I : (\forall \alpha < \beth_{(2^\mu)^+}) t_\alpha^* < t\}$, so every $A \subseteq I_1$ of cardinality $\leq \beth_{(2^\mu)^+}$ has a bound from below, so for some $q_1 \in \mathscr{S}^\alpha(M)$ the set $I_2 = \{t \in I_1 : \text{tp}(\bar{a}_t {}^\frown \bar{b}, \emptyset, N) = q_1\}$ is unbounded from below in I_1. If $q_1 \neq p$ then q_1, p can serve as p_1, p_2, so assume $q_1 = p$, so q, q_1 can serve as p_1, p_2.]

Now we apply 4.6(5) with h_i^θ listing $\bar{a}_\alpha^\theta {}^\frown \bar{a}_{\theta+\alpha}^\theta {}^\frown \bar{b}$ and letting N^1 be $EM(I, \Phi)$ (so $\tau(N^1) = \tau(\Phi)$) and we get Ψ as there. Now for any linear order I^*, look at $EM(I^*, \Psi)$ and its skeleton $\langle \bar{a}_t^* : t \in I^* \rangle$. Clearly $\bar{a}_t^* = \bar{a}_t^1 {}^\frown \bar{a}_t^2 {}^\frown \bar{b}^*$, and letting M^* be the submodel of $EM_{\tau(\Phi)}(I^*, \Psi)$ generated by $\{a_t^1, a_t^2 : t \in I^*\} \cup \bar{b}$, it is isomorphic to $EM(I^* + I^*, \Psi)$, so without loss of generality $M = M^* \restriction \tau(\mathfrak{K}) \leq_{\mathfrak{K}} \mathfrak{C}$, so $\text{tp}(\bar{a}_t^1 {}^\frown \bar{b}, \emptyset, M) = p_1, \text{tp}(\bar{a}_t^2 {}^\frown \bar{b}, \emptyset, M) = p_2$. Now for any χ we can choose $I^* = I_\chi^*$ such that $\mathbf{D} = \{J : J \text{ an initial segment of } I^* \text{ and } J \cong I^* \text{ and } I^* \backslash J \text{ is isomorphic to } I^*\}$ has cardinality $> \chi$.

So we have proved clause (E) and clause (B), by easy manipulations we get clause (A) and so (C).

We are left with clause (D). Clearly there is $\bar{t} = \langle t_i : i < i^* \rangle$ satisfying $i^* < |\alpha|^+ + \aleph_0$ such that $\bar{b} = \langle b_\beta : \beta < \alpha \rangle, b_\beta = \tau_\beta(\bar{a}_{t_{i(\beta,0)}}, \ldots, \bar{a}_{t_{i(\beta,n(\beta)-1)}})$ where $i(\beta, \ell) < i^*, \tau_\beta$ a $\tau(\Phi)$-term.

Let $J = \{t_i : i < i^*\}$ so by the version of (∗) used in clause (D), necessarily for some $s_1, s_2 \in I \backslash J$ we have:

$$p_1 \neq p_2 \text{ where}$$

$$p_1 = \ \text{tp}(\bar{a}_{s_1} {}^\frown \bar{b}, \emptyset, N)$$

$$p_2 = \ \text{tp}(\bar{a}_{s_2} {}^\frown \bar{b}, \emptyset, N)$$

Clearly $s_1 \neq s_2$. By renaming without loss of generality $s_1 <^I s_2$ and $0 = i_0 \leq i_1 \leq i_2 \leq i_3 = i^*$ and $t_i <^I s_1 \Leftrightarrow i < i_1$ and $s_1 <^I t_i <^I s_2 \Leftrightarrow i_1 \leq i < i_2$ and $s_2 <^I t_i \Leftrightarrow i_2 < i < i_3$.

As I is $(\beth_{(2^\mu)^+})^+$-strongly saturated we can increase J so renaming without loss of generality $i(\beta, \ell) \notin \{i_1, i_2\}$, and replace t_{i_1}, t_{i_2} by s_1, s_2. So for every linear order I' we can define a linear order I^* with a set of elements

$$\{t_i : i < i_1 \text{ or } i_2 < i < i^*\} \cup \{(s, i) : s \in I', i_1 \leq i < i_2\}$$

linearly ordered by:

$$t_{j_1} < t_{j_2} \ \underline{\text{if}} \ j_1 < j_2 < i_1$$

$$t_{j_1} < t_{j_2} \ \underline{\text{if}} \ i_2 < j_1 < j_2 < i^*$$

$$t_{j_1} < (s', j') < (s'', j'') < t_{j_2} \ \underline{\text{if}} \ j_1 < i_1, i_2 < j_2 < i^*,$$
$$s', s'' \in I', j', j'' \in [i_1, i_2]$$
$$(s' <^{I'} s'') \vee (s' = s'' \ \& \ j' < j'').$$

In $M = EM(I^*, \Phi)$ define, for $s \in I'$

$$\bar{c}_{s,i} \text{ is } \bar{a}_{t_i} \text{ if } i < i_1 \vee i > i_2,$$

$$\bar{c}_{s,i} = \bar{a}_{(s,i)} \text{ if } i \in [i_1, i_2]$$

$$\bar{b}_s = \langle \tau_\beta(\bar{c}_{s,i(\beta,0)}, \bar{c}_{s,i(\beta,1)}, \ldots, \bar{c}_{s,i(\beta,n(\beta)-1)}) : \beta < \alpha \rangle.$$

Easily

$$s' <^{I'} s'' \Rightarrow \ \text{tp}(\bar{a}_{(s',i_1)}\hat{\ }\bar{b}_{s''}, \emptyset, M) = p_1$$

$$s'' \le^{I'} s' \Rightarrow \ \text{tp}(\bar{a}_{(s',i_1)}\hat{\ }\bar{b}_{s''}, \emptyset, M) = p_2.$$

By easy manipulations we can finish. $\hfill\square_{4.15}$

4.16 Claim. *Assume K is categorical in λ and*

(a) $1 \le \kappa$ *and* $LS(\mathfrak{K}) < \theta = \ \text{cf}(\theta) \le \lambda$ *and*
$(\forall \alpha < \theta)(|\alpha|^\kappa < \theta)$

(b) $\bar{a}_i \in {}^\kappa\mathfrak{C}$ *for* $i < \theta$.

<u>Then</u> *for some $W \subseteq \theta$ of cardinality θ, the sequence $\langle \bar{a}_i : i \in W \rangle$ is strictly indiscernible.*

Proof of 4.16. Let $M' \prec \mathfrak{C}, \|M'\| = \theta$ and $\alpha < \theta \Rightarrow \bar{a}_\alpha \subseteq M'$. There is $M'', M' \prec M'' \prec \mathfrak{C}, \|M''\| = \lambda$. So $M'' \cong EM(\lambda, \Phi)$ and without loss of generality equality holds. So there is $u \subseteq \lambda, |u| \le \theta$ such that $M' \subseteq EM(u, \Phi)$. So without loss of generality $M' = EM(u, \Phi)$. So $a_\alpha \in EM(v_\alpha, \Phi)$ for some $v_\alpha \subseteq u, |v_\alpha| \le \kappa$.
Without loss of generality: $\text{otp}(v_\alpha) = j^*$, so for $\alpha < \beta$, $\text{OP}_{u_\alpha, u_\beta}$ the order preserving map from v_β onto v_α induces $f_{\alpha,\beta} : EM(u_\beta, \Phi) \xrightarrow[\text{onto}]{\text{iso}} EM(u_\alpha, \Phi)$, and without loss of generality $f_{\alpha,\beta}(\bar{a}_\beta) = \bar{a}_\alpha$.

Now as u is well ordered and the assumption (a), (or see below) for some $w \in [\theta]^\theta$ the sequence $\langle v_\alpha : \alpha \in w \rangle$ <u>is indiscernible</u> in the linear order sense (make them a sequence). Now we can create the right Φ.

[Why? Let $u_\alpha = \{\gamma_{\alpha,j} : j < j^*\}$ where $\gamma_{\alpha,j}$ increases with j. For $\alpha < \theta$, let
$A_\alpha = \{\gamma_{\beta,j} : \beta < \alpha, j < j^*\} \cup \{\bigcup_{\beta < \alpha, j} \gamma_{\beta,j} + 1\}$. Let $\gamma^*_{\beta,j} = \text{Min}\{\gamma \in A_\alpha : \gamma_{\beta,j} \geq \gamma\}$
and for each $\alpha \in S_0^* = \{\delta < \theta : \text{cf}(\delta) > \kappa\}$ let $h(\delta) = \text{Min}\{\beta < \delta : \gamma^*_{\delta,j} \in A_\beta\}$
(defining $\langle A_\beta : \beta \leq \delta \rangle$ as increasing continuous, $\text{cf}(\delta) > \kappa \geq |j^*|$ and $\gamma^*_{\delta,j} \in A_\delta$ by definition).

By Fodor's lemma for some stationary $S_1 \subseteq S_0$, $h \restriction S_1$ is constantly β^*. As $(\forall \alpha < \theta)(|\alpha|^\kappa < \theta = \text{cf}(\theta))$ for some $S_2 \subseteq S_1$ for each $j < j^*$ and for all $\delta \in S_2$, the truth value of "$\gamma_{\delta,j} \in A_\delta$" (e.g. $\gamma_{\delta,j} = \gamma^*_{\delta,j}$) is the same and $\langle \gamma^*_{\delta,j} : \delta \in S_2 \rangle$ is constant. Now $\langle u_\delta : \delta \in S_2 \rangle$ is as required. See more [Sh 620, §7].] $\square_{4.16}$

4.17 Definition. A model M is λ-strongly saturated if:

(a) λ-saturated

(b) strongly λ-homogeneous: if f is a partial elementary mapping from M to M, $|\text{Dom}(f)| < \lambda$
then $(\exists g \in AUT(M))(f \subseteq g)$.

Note: if $\mu = \mu^{<\lambda}$, I a linear order of cardinality $\leq \mu$, then there is a λ-strongly saturated dense linear order $J, I \subseteq J$.

Remark. We can even get a uniform bound on $|J|$ (which only depends on μ).

§5 Rank and Superstability

5.1 Definition. For $M \in K_\mu, p \in \mathscr{S}^m(M)$ we define $R(p)$ an ordinal or ∞ as follows: $R(p) \geq \alpha$ iff for every $\beta < \alpha$ there are $M^+, M \leq_\mathfrak{K} M^+ \in K_\mu, p \subseteq p^+ \in \mathscr{S}^1(M^+), R(p^+) \geq \beta$ & $[p^+ \ \mu\text{-strongly splits over } M]$. In case of doubt we write R_μ. This is well defined and has the obvious properties:

- (a) monotonicity,
- (b) if $M \in K_\mu, p \in \mathscr{S}^m(M)$ and $\mathrm{Rk}(p) \geq \alpha$ <u>then</u> for some N, q satisfying $M \leq_\mathfrak{K} N \in K_\mu$ and $q \in \mathscr{S}^m(N)$ we have: $q \upharpoonright M = p$ and $\mathrm{Rk}(q) = \alpha$
- (c) automorphisms of $\mathfrak{C}$ preserve everything
- (d) the set of values is $[0, \alpha)$ or $[0, \alpha) \cup \{\infty\}$ for some $\alpha < (2^\mu)^+$, etc.

5.2 Definition. We say $\mathfrak{K}$ is $(\mu, 1)$-superstable if
$$M \in K_\mu \ \& \ p \in \mathscr{S}(M) \Rightarrow R(p) < \infty \ \left(\text{equivalently } < (2^\mu)^+\right).$$

5.3 Claim. *If $(*)_\mu$ from 4.13 above fails, <u>then</u> $(\mu, 1)$-superstability fails.*

Proof. Straight.

5.4 Claim. *If $\mathfrak{K}$ is not $(\mu, 1)$-superstable, <u>then</u> there are a sequence $\langle M_i : i \leq \omega + 1 \rangle$ which is $<_\mathfrak{K}$-increasing continuous in K_μ and $m < \omega$ and $\bar{a} \in {}^m(M_{\omega+1})$ such that $(\forall i < \omega)\left[\frac{\bar{a}}{M_{i+1}} \text{ does } \mu\text{-strongly split over } M_i\right]$. Also the inverse holds.*

Proof. As usual.

5.5 Claim. *1) If $\mathfrak{K}$ is not $(\mu, 1)$-superstable <u>then</u> K is unstable in every χ such that $\chi^{\aleph_0} > \chi + \mu + 2^{\aleph_0}$.*
2) If $\kappa \in \kappa_\mu^*(\mathfrak{K})$ and $\chi^\kappa > \chi \geq LS(\mathfrak{K})$, <u>then</u> $\mathfrak{K}$ is not χ-stable, even modulo E_μ.
3) If $\kappa \in \kappa_\mu(\mathfrak{K})$ and $\chi^\kappa > \chi = \chi^\kappa \geq LS(\mathfrak{K})$ or just there is a tree with χ nodes and $> \chi \ \kappa$-branches and $\chi \geq LS(\mathfrak{K})$, <u>then</u> $\mathfrak{K}$ is not χ stable even modulo E_μ.

Remark. We intend to deal with the following elsewhere; we need stable amalgamation

- $(*)$ if $\kappa \in \kappa_\mu(\mathfrak{K})$, $\mathrm{cf}(\chi) = \kappa$, $\bigwedge\limits_{\lambda < \chi} \lambda^\mu \leq \chi$,

    <u>then</u> $\mathfrak{K}$ is not χ-stable.

5.6 Remark. 1) In (1) this implies $I(LS(\mathfrak{K})^{+(\omega(\alpha_0 + \alpha) + n)}, K) \geq |\alpha|$ when $\mu = \aleph_{\alpha_0}$. We conjecture that [GrSh 238] can be generalized to the content of (1) with cardinals which exists by ZFC.

2) Note that for FO stable theory $T, \mathfrak{K} = \mathrm{MOD}(T)$, for κ regular we have $(*)_1^\kappa \Leftrightarrow$ $(*)_2^\kappa$ where

$(*)_1^\kappa$ for any increasing chain $\langle M_i : i < \kappa \rangle$ of λ-saturated models of length κ, the union $\bigcup_{i<\kappa} M_i$ is λ-saturated,

$(*)_2^\kappa$ $\kappa \in \kappa_r(\mathfrak{K})$.

In [Sh:e], $(*)_2^\kappa$ is changed to

$(**)$ $\kappa < \kappa_r(T)$
(really $\kappa_r(\mathfrak{K})$ (i.e. $\kappa_r(T)$) is a <u>set</u> of regular cardinals)).

From this point of view, FO theory T is a degenerated case: $\kappa_r(T)$ is an initial segment so naturally we write the first regular not in it. This is a point where [Sh 300] opens our eyes.

3) In fact in 5.5 not only do we get $\|M\| = \chi, |\mathscr{S}(M)| > \chi$ but also $|\mathscr{S}(M)/E_\mu| > \chi$.

4) Let me try to explain the proof of 5.5, of course, being influenced by the first order case. If the class is superstable, one of the consequences of not having the appropriate order property is that (see 4.15) for a strictly indiscernible sequence $\langle \bar{a}_t : t \in I \rangle$ over A each $\bar{a}_t$ of length at most μ and $\bar{b}$, singleton for simplicity, for all except few of the $\bar{a}_t$'s, the type of $\bar{a}_t \cong \bar{b}$ realizes the same type. Of course, we can get better theorems generalizing the ones for first order theories: we can use $\kappa \notin \kappa_\mu(\mathfrak{C})$ and/or demand that after adding to $A, \bar{c}$ and few of the $\bar{a}_t$'s the rest is strictly indiscernible over the new A, but this is not used in 5.5. Now if $\mathfrak{C}$ is $(\mu, 1)$-superstable the number of exceptions is finite, however, the inverse is not true: for some non $(\mu, 1)$-superstable class $\mathfrak{C}$ still the number of exceptions in such situations is finite. In the proof of 5.5(1) this property is used as a dividing line.

Proof. 1)

Case I There are $M, N, p, \langle \bar{a}_i : i < i^* \rangle$ as in $4.13(*)_\mu$ and $\bar{c}$, (in fact $\ell g(\bar{c}) = 1$) such that $\bar{c}$ realizes $h_i(p)$ for infinitely many i's and fails to realize $h_i(p)$ for infinitely many i's.

Let I be a $\beth(\chi + \beth_{(2^\mu)^+})^+$-strongly saturated dense linear order (see Definition 4.17) such that even if we omit $\leq \beth_{(2^\mu)^+}$ members, it remains so. By the strict indiscernibility we can find $\langle \bar{a}_t : t \in I \rangle, c$ as above.

So there is $u \subseteq I, |u| < \beth_{(2^\mu)^+}$ such that $q = \mathrm{tp}(\bar{a}_t \hat{\ } \bar{c}, \emptyset, \mathfrak{C})$ is the same for all $t \in I \backslash u$; without loss of generality $q = \mathrm{tp}(\bar{a}_t \hat{\ } \bar{c}, \emptyset, \mathfrak{C}) \Leftrightarrow t \in I \backslash u$, so u is infinite. So we can find $i_n \in i^* \cap u$ such that $i_n < i_{n+1}$. Let $I' = I \backslash (u \backslash \{i_n : n < \omega\})$, so that I' is still χ^+-strongly saturated. Hence for every $J \subseteq I'$ of order type ω for some $c_J (\in \mathfrak{C})$ we have

$$t \in I' \backslash J \Rightarrow \mathrm{tp}(\bar{a}_t \hat{\ } \bar{c}_J, \emptyset, \mathfrak{C}) = q$$

$$t \in J \Rightarrow \mathrm{tp}(\bar{a}_t \hat{\ } \bar{c}_J, \emptyset, \mathfrak{C}) \neq q.$$

This clearly suffices.

Case II Not Case I.

As in [Sh 3] (the finitely many finite exceptions do not matter) or see part (2).

2) If $\chi < 2^\kappa$ the conclusion follows from 3.3(2). Possibly decreasing κ (allowable as $\kappa \in \kappa_\mu^*(\mathfrak{K})$ rather than $\kappa \in \kappa_\mu(\mathfrak{K})$ is assumed) we can find a tree $\mathscr{T} \subseteq {}^{\kappa \geq}\chi$, so closed under initial segments such that $|\mathscr{T} \cap {}^{\kappa >}\chi| \leq \chi$ but $|\mathscr{T} \cap {}^\kappa\chi| > \chi$. (The cardinal arithmetic assumption is needed just for this). Let $\langle M_i : i \leq \kappa + 1\rangle, c \in M_{\kappa+1}$ exemplify $\kappa \in \kappa_\mu^*(\mathfrak{K})$ and let $\mathscr{T}' = \mathscr{T} \cup \{\eta^\frown\langle 0\rangle : \eta \in {}^\kappa\mathrm{Ord}$ and $i < \kappa \Rightarrow \eta \restriction i \in \mathscr{T}\}$.

Now we can by induction on $i \leq \kappa + 1$ choose $\langle h_\eta : \eta \in \mathscr{T}' \cap {}^i\chi\rangle$, such that:

 (a) h_η is a $\leq_\mathfrak{K}$-embedding from $M_{\ell g(\eta)}$ into $\mathfrak{C}$

 (b) $j < \ell g(\eta) \Rightarrow h_{\eta \restriction j} \subseteq h_\eta$

 (c) if $i = j+1, \nu \in \mathscr{T} \cap {}^j\chi$, <u>then</u> $\langle h_\eta(M_i) : \eta \in \mathrm{Suc}_T(\nu)\rangle$ is strictly indiscernible, and can be extended to a sequence of length $\bar{\kappa}$ such that $\langle h_\eta(p \restriction M_i) : \eta \in \mathrm{Suc}_I(\nu)\rangle$ is contradictory (i.e. as in Definition 4.9(1)).

There is no problem to do this. Let $M \leq_\mathfrak{K} \mathfrak{C}$ be of cardinality χ and include $\bigcup\{h_\eta(M_i) : i < \kappa$ and $\eta \in \mathscr{T} \cap {}^i\chi\}$ hence it includes also $h_\eta(M_\kappa)$ if $\eta \in \mathscr{T} \cap {}^\kappa\chi$ as $M_\kappa = \bigcup_{i<\kappa} M_i$.

For $\eta \in \mathscr{T} \cap {}^\kappa\chi$ let $c_\eta = h_{\eta^\frown<0>}(c)$ and $M_\eta = h_\eta(M_i)$ when $\eta \in \mathscr{T} \cap {}^i\mathrm{Ord}$ and $i \leq \kappa + 1$, so by 4.15 clearly (by clause (C))

 (∗) if $i < \kappa, \eta \in \mathscr{T} \cap {}^i\chi$, and $\eta \triangleleft \eta_1 \in \mathscr{T} \cap {}^\kappa\chi$, <u>then</u>
       $\{\rho \in \mathrm{Suc}_T(\eta) :$ for some $\rho_1, \rho \triangleleft \rho_1 \in \mathscr{T} \cap {}^\kappa\chi$ and
               c_{ρ_1} realizes $\mathrm{tp}(c_{\eta_1}, h_{\eta_1 \restriction (i+1)}(m_{i+1}))\}$
      has cardinality $< \beth_{(2^\mu + LS(\mathfrak{K}))^+}$.

Next define an equivalence relation **e** on $\mathscr{T} \cap {}^\kappa\chi$:

$$\eta_1 \, \mathbf{e} \, \eta_2 \text{ iff } \mathrm{tp}(c_{\eta_1}, M) = \mathrm{tp}(c_{\eta_2}, M).$$

or just

$$\eta_1 \mathbf{e} \eta_2 \text{ iff } (\forall\nu)[\nu \in \mathscr{T} \Rightarrow \mathrm{tp}(c_{\eta_1}, M_\nu) = \mathrm{tp}(c_{\eta_2}, M_\nu)].$$

Now if for some $\eta \in \mathscr{T} \cap {}^\kappa\chi, |\eta/\mathbf{e}| > \beth_{(2^\mu + LS(\mathfrak{K}))^+}$ then for some $\eta^* \in \mathscr{T} \cap {}^{\kappa >}\chi$, we have

$$\{\nu \restriction (\ell g(\eta^* + 1)) : \nu \in \eta/\mathbf{e}\} \text{ has cardinality } > \beth_{(2^\mu + LS(\mathfrak{K}))^+}$$

which contradicts (∗); so if $\chi \geq \beth_{(2^\mu + LS(\mathfrak{K}))^+}$, we are done.

   But if for some $\eta \in \mathscr{T} \cap {}^{\kappa >}\chi$ the set in (∗) has cardinality $\geq \kappa$, then we can continue as in case I of the proof of part (1) replacing "infinite" by "of cardinality $\geq \kappa$", so assume this never happens. So above if $|\eta/\mathbf{e}| > 2^\kappa$, we get again a contradiction. So if $|\mathscr{T} \cap {}^\kappa\chi| > 2^\kappa$, we conclude $|\mathscr{T} \cap {}^\kappa\chi/\mathbf{e}| = |\mathscr{T} \cap {}^\kappa\chi|$, so we are done. We are left with the case $\chi < 2^\kappa$, covered in the beginning (note that for $\chi < 2^\kappa$ the interesting notion is splitting).

3) Proof similar to part (2). $\square_{5.5}$

5.7 Claim. *If $\lambda > \mu^+, \mu \geq LS(\mathfrak{K}, K)$, $\mathfrak{K}$ is categorical in λ <u>then</u>*
1) K is $(\mu, 1)$-superstable.
2) $\kappa_\mu^*(\mathfrak{K})$ is empty.

Proof. 1) Assume the conclusion fails. If $\lambda > \mu^{+\omega}$, we can use 5.5 + 1.7 so without loss of generality $\mathrm{cf}(\lambda) > LS(\mathfrak{K})$.

By 1.7 if $M \in K_\lambda$ then M is $\mathrm{cf}(\lambda)$-saturated. On the other hand from the Definition of $(\mu, 1)$-superstable we get a non-μ^+-saturated model.

Let $\chi = \beth_{(2^\lambda)^+}$. Assume $\mathfrak{K}$ is not $(\mu, 1)$- superstable so we can find in K_μ an increasing continuous sequence $\langle M_i : i \leq \kappa + 1 \rangle$ and $c \in M_{\omega+1}$ such that $p_{n+1} = \mathrm{tp}(c, M_{n+1}, M_{\omega+1})$ μ-strongly splits over M_n for $n < \omega$. For each $n < \omega$ let $\langle \bar{a}_i^n : i < \omega \rangle$ be a strictly indiscernible sequence over M_n exemplifying p_{n+1} does μ-strongly splits over M_n (see Definition 4.12). So we can define $\bar{a}_i^n \in \mathfrak{C}$ for $i \in [\omega, \chi)$ such that $\langle \bar{a}_i^n : i < \chi \rangle$ is strictly indiscernible over M_n. Let $\mathscr{T}_n = \{\eta \in {}^{2n}\chi : \eta(2m) < \eta(2m+1)$ for $m < n\}$. For $n < \omega, i < j < \chi$ let $h_{i,j}^n \in \mathrm{AUT}(\mathfrak{C})$ be such that $h_{i,j}^n \upharpoonright M_n = \mathrm{id}, h_{i,j}^n(\bar{a}_0^n {}^\frown \bar{a}_1^n) = \bar{a}_i^n {}^\frown \bar{a}_j^n$. Now we choose by induction on $n < \omega, \langle f_\eta : \eta \in \mathscr{T}_n \rangle, \langle g_\eta : \eta \in \mathscr{T}_n \rangle, \langle a_i^n : i < \chi, \eta \in \mathscr{T}_n \rangle$ such that:

(a) f_η, g_η are restrictions of automorphisms of $\mathfrak{C}$

(b) $\mathrm{Dom}(f_\eta) = M_n$

(c) $g_\eta \in \mathrm{AUT}(\mathfrak{C})$

(d) $\bar{a}_i^n = g_\eta(\bar{a}_i^n)$ if $\eta \in T_n$

(e) $f_{<>} = \mathrm{id}_{M_0}$,

(f) $f_\eta \subseteq g_\eta$

(g) if $\eta \in {}^{2n}\chi, m < n$ then $f_{\eta \upharpoonright (2m)} \subseteq f_\eta$

(h) if $\eta \in {}^{2n}\chi$ and $i < j < \chi$ then $f_{\eta^\frown <i,j>} = (g_\eta \circ h_{i,j}^n) \upharpoonright M_{n+1}$.

There is no problem to carry the induction. Now choose by induction on $n, M_n^*, \eta_n, i_n, j_n$ such that

(α) $i_n < j_n < \chi$ and $\eta_n = \langle i_0, j_0, \ldots, i_{n-1}, j_{n-1} \rangle$ so $\eta_n \in \mathscr{T}_n$

(β) $M_n \in K_\lambda, M_n^* <_{\mu,\omega}^1 M_{n+1}^*$

(γ) $\mathrm{Rang}(f_{\eta_n}) \subseteq M_n$

(δ) $\bar{a}_i^{\eta_n}, \bar{a}_j^{\eta_n}$ realizes the same type over M_n

(ε) $\bar{a}_i^{\eta_n}, \bar{a}_j^{\eta_n} \subseteq M_{n+1}^*$.

There is no problem to carry the induction (using the theorem on existence of strictly indiscernibles to choose $i_n < j_n$).

So $\bigcup_{n<\omega} f_{\eta_n}$ can be extended to $f \in \mathrm{AUT}(\mathfrak{C})$. Let $c^* = f(c), M_\omega^* = \bigcup_n M_{\eta_n}^*, M_{\omega+1}^* = f(M_{\omega+1})$. Clearly $\mathrm{tp}(c, M_{n+1}^*, M_{\omega+1}^*)$ does μ-split over M_n hence M_ω is not μ^+-saturated (as $\mathrm{cf}(\lambda) > \mu$) (see 5.8); contradiction.
2) Follows. $\square_{5.7}$

5.8 Claim. *If $\mu \geq LS(\mathfrak{K}), \langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous,*
$p \in \mathscr{S}^{\leq \mu}(M_\delta), p$ μ-strongly splits over M_i for all i (or just μ-splits over M_i) and
$\delta < \mu^+$ <u>then</u> M_δ is not μ^+-saturated.

Proof. Straight.

5.9 Claim. *Assume there is a Ramsey cardinal $> \mu + LS(\mathfrak{K})$. If $\mathfrak{K}$ is not $(\mu, 1)$-*
superstable, <u>then</u> for every $\chi > \mu + LS(\mathfrak{K})$ there are 2^χ pairwise non-isomorphic
models in $\mathfrak{K}_\chi$.

Proof. By [GrSh 238] for χ regular; together with [Sh:e] for all χ.

5.10 Lemma. *1) If for some $M, |\mathscr{S}(M)/E_\mu| > \chi \geq \|M\| + \beth_{(2^\mu)^+}$ and $\mu \geq LS(\mathfrak{K})$*
<u>then</u> $\mathfrak{K}$ is not $(\mu, 1)$-superstable.
2) If $\chi^\kappa \geq |\mathscr{S}(M)/E_\mu| > \chi^{<\kappa} \geq \chi \geq \|M\| + \beth_{(2^\mu)^+}, \mu \geq LS(\mathfrak{K}) + \kappa$ <u>then</u> $\kappa \in \kappa^*_\mu(\mathfrak{K})$.

Proof. No new point when you remember the definition of E_μ (see 1.8).

§6 Existence of Many Non-Splitting

6.1 Question. Suppose $\kappa + LS(\mathfrak{K}) \leq \mu < \lambda$ and $\bar{N} = \langle N_i : i \leq \delta \rangle$ is $<^1_{\mu,\kappa}$-increasing continuous (we mean for $i < j, j$ non-limit $N_i <^1_{\mu,\kappa} N_j$), $\delta < \mu^+$ and $p \in \mathscr{S}^m(N_\delta)$. Is there $\alpha < \delta$ such that for every $M \in \mathfrak{K}_{\leq\lambda}, N_\delta \leq_{\mathfrak{K}} M, p$ has an extension $q \in \mathscr{S}^m(M)$ which does not μ-split over N_α (and so in particular p does not μ-split over N_α).

6.2 Remark. If $p \restriction N_{\alpha+1}$ does not μ-split over N_α, then $p \restriction N_{\alpha+1}$ has at most one extension mod E_μ which does not μ-split over N_α because $N_{\alpha+1} \in K_\mu$ is universal over $N_\alpha, N_{\alpha+1} \leq_{\mathfrak{K}} M \in K_\lambda$. So in 6.1 if p does not μ-split over N_α, <u>then</u> there is at most one q/E_μ.

6.3 Lemma. *Suppose K is categorical in $\lambda, \mathrm{cf}(\lambda) > \mu \geq LS(\mathfrak{K})$. <u>Then</u> the answer to question 6.1 is yes.*

6.4 Remark. We intend later to deal with the case $\lambda > \mu \geq \mathrm{cf}(\lambda) + LS(\mathfrak{K})$ as in [KlSh 362].

Notation. $I \times \alpha$ is $I + I + \ldots (\alpha$ times) (with the obvious meaning).

Proof. Let Φ be proper for linear order, $|\tau(\Phi)| \leq LS(\mathfrak{K}), EM_\tau(I, \Phi) \in K$ (of power $|I| + \mu(K)$) where I is a linear order, of course and $I \subseteq J \Rightarrow EM_\tau(I, \Phi) \leq_{\mathfrak{K}} EM_\tau(J, \Phi)$. So $EM_\tau(\lambda, \Phi)$ is μ^+-saturated (by 1.7). Let I^* be a linear order of power μ such that $I^* \times (\alpha + 1) \cong I^*$ for $\alpha < \mu^+$ and $I^* \times \omega \cong I^*$. By 1.7 we know that $EM_\tau(I^* \times \lambda, \Phi)$ is μ^+-saturated.

Now we choose by induction on i an ordinal $\alpha_i < \mu^+$ and an isomorphism h_i from N_{1+i} onto $EM(I^* \times \alpha_i, \Phi)$, both increasing with i where N_{i+1} is from 6.1 and $\mathrm{cf}(\alpha_i) = \aleph_0$ for i nonlimit.

For $i = 0$, use the proof of the uniqueness of N_1 over N_0 (see 2.6 and reference there); more specifically using the back and forth argument we can find $J_0 \subseteq \lambda, |J_0| = \mu$ and isomorphism h_0 from $N_1 = N_{0+1}$ onto $EM(I^* \times J_0, \Phi) \subseteq (I^* \times \lambda, \Phi)$. Now let $J^0 = J_0 \cup \{\alpha < \lambda : (\forall\beta \in J_0)\, \beta < \alpha\}$ so $J^0 \cong \lambda$ (note: J_0 is bounded in λ as $\mathrm{cf}(\lambda) > \mu \geq |J_0|$) and also $EM_\tau(I^* \times J^0, \Phi)$ is μ^+-saturated (being isomorphic to $EM_\tau(I^* \times \lambda, \Phi)$), so without loss of generality J_0 is some ordinal $\alpha_0 < \mu^+$.

So we have h_0. The continuation is similar.

Now h_δ is defined $h_\delta : N_\delta \overset{\text{onto}}{\to} EM_\tau(I^* \times \alpha_\delta, \Phi)$, so as $EM_\tau(I^* \times \lambda, \Phi)$ is μ^+-saturated, $h_\delta(p)$ is realized say by $\bar{a}$, so let $\bar{a} = \bar{\sigma}(x_{(t_1,\gamma_1)}, \ldots, x_{(t_n,\gamma_n)})$ where $\bar{\sigma}$ is a sequence of terms in $\tau(\Phi)$ and (t_ℓ, γ_ℓ) is increasing with ℓ (in $I^* \times \lambda$). Let $\beta < \delta$ be such that:

$$\{\gamma_1, \ldots, \gamma_n\} \cap \alpha_\delta \subseteq \alpha_\beta.$$

Let

$$\gamma'_\ell = \begin{cases} \gamma_\ell & \underline{\text{if}} \quad \gamma_\ell < \alpha_\delta \\ \lambda + \gamma_\ell & \underline{\text{if}} \quad \gamma_\ell \geq \alpha_\delta \end{cases}$$

Then in the model $N = EM_{\tau(\mathfrak{K})}(I^* \times \lambda + \lambda, \Phi)$, we shall show that the finite sequence $\bar{\sigma}^1 = \bar{\sigma}\big(x_{(t_1, \gamma'_1)}, \ldots, x_{(t_n \gamma'_n)}\big)$ realizes a type as required over $M = EM_{\tau(\mathfrak{K})}(I^* \times \lambda, \Phi)$. Why? Let $M_\gamma = EM_{\tau(\mathfrak{K})}(I^* \times \alpha_\gamma, \Phi)$ for $\gamma < \delta$. Assume toward contradiction that

(∗) $\text{tp}(\bar{a}', M, N)$ does μ-split over $M_{\beta+1}$.

Let $\bar{c}, \bar{b} \in {}^\mu M$ realize the same type over $M_{\beta+1}$ but witness splitting.

We can find $w \subseteq \lambda, |w| \leq \mu$ such that $\bar{c}, \bar{b} \subseteq EM(I^* \times w, \Phi)$. Choose γ such that

$$\sup(w) < \gamma < \lambda.$$

Let $M^- = EM_{\tau(\mathfrak{K})}(I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda)), \Phi) <_{\mathfrak{K}} M$.

Let $N^- = EM_{\tau(\mathfrak{K})}(I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda) \cup [\lambda, \lambda + \lambda)), \Phi) <_{\mathfrak{K}} N$.

So still $\bar{c}, \bar{b}$ witness that $\text{tp}(\bar{a}', M^-, N^-)$ does μ-split over $M_{\beta+1}$.

There is an automorphism f of the linear order $I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda)) \cup [\lambda, \lambda + \lambda))$ such that

$$f \restriction (I^* \times \alpha_{\beta+1}) = \text{ the identity}$$

$$f \restriction (I^* \times [\gamma + 1, \lambda + \lambda)) = \text{ the identity}$$

$$\text{Rang}\big(f \restriction (I^* \times w)\big) \subseteq I^* \times [\alpha_{\beta+1}, \alpha_{\beta+2}).$$

Now f induces an automorphism of N^- naturally called $\hat{f}$.
So

$$\hat{f} \restriction M_\beta = \text{ identity}$$

$$\hat{f}(\bar{a}') = \bar{a}'$$

$$\hat{f}(M^-) = M^-$$

As $\hat{f}$ is an automorphism, $\hat{f}(\bar{c}), \hat{f}(\bar{b})$ witness that $\text{tp}(\hat{f}(\bar{a}'), \hat{f}(M^-), \hat{f}(N^-))$ does μ-splits over $\hat{f}(M_{\alpha_{\beta+1}})$; i.e. $\text{tp}(\bar{a}', M^-, N^-)$ does μ-splits over $M_{\alpha_{\beta+1}}$. So $\text{tp}(\bar{a}', M_{\alpha_{\beta+2}}, N)$ does μ-splits over $M_{\alpha_{\beta+1}}$.

Now choose $\alpha_\gamma < \mu^+$ for $\gamma \in (\delta, \mu^+]$, increasing continuous by

$$\alpha_{\delta+i} = \alpha_\delta + i$$

$$M_\gamma = EM_{\tau(\mathfrak{K})}(I^* \times \alpha_\gamma, \Phi).$$

So $\langle M_\gamma : \gamma \leq \mu \rangle$ is increasing continuous. So for $\gamma_1 \in [\beta, \mu^+)$ there is $f \in AUT(I^* \times (\lambda + \lambda))$ such that

$$f \upharpoonright I^* \times \alpha_\beta = \text{ identity}$$

$$f \text{ takes } I^* \times [\alpha_\beta, \alpha_{\beta+1}) \text{ onto } I^* \times [\alpha_\beta, \alpha_{\gamma_1+1})$$

$$f \text{ takes } I^* \times [\alpha_{\beta+1}, \alpha_{\beta+2}) \text{ onto } I^* \times \{\alpha_{\gamma_1+1}\}$$

$$f \text{ takes } I^* \times [\alpha_{\beta+2}, \alpha_{\gamma_1+2}) \text{ onto } I^* \times \{\alpha_{\gamma_1+2}\}$$

$$f \upharpoonright I^* \times [\alpha_{\gamma_1+2}, \lambda + \lambda) = \text{ identity}.$$

As before this shows (using obvious monotonicity of μ-splitting)

$$\text{tp}(\bar{a}^1, M_{\gamma_1+2}N) \, \mu\text{-splits over } M_{\gamma_1+1}.$$

So $\{\gamma < \mu : \text{tp}(\bar{a}', M_{\gamma+1}, N) \text{ does } \mu\text{-split over } M_\gamma\}$ has order type μ, so without loss of generality is μ. By 3.3(2) we get a contradiction. $\qquad \square_{6.3}$

6.5 Theorem. *Suppose K categorical in λ and the model in K_λ is μ^+-saturated (e.g. $\text{cf}(\lambda) > \mu$) and $LS(\mathfrak{K}) \leq \mu < \lambda$.*
1) $M <^1_{\mu,\kappa} N \Rightarrow N$ is saturated if $LS(\mathfrak{K}) < \mu$.
2) If κ_1, κ_2 and for $\ell = 1, 2$ we have $M_\ell <^1_{\mu,\kappa_\ell} N_\ell$, <u>then</u> $N_1 \cong N_2$.
3) There is $M \in K_\mu$ which is saturated.

6.6 Remark. 1) The model we get by (2) we call **the saturated model** of $\mathfrak{K}$ in μ.
2) Formally — we do not use 6.3.
3) By the same proof $M \leq^1_{\mu,\kappa_\ell} N_\ell \Rightarrow N_1 \cong_M N_2$ and we call N **saturated over** M.

Proof. 1) By the uniqueness proofs 2.2 as $M <^1_{\mu,\kappa} N$ there are $\langle M_i : i \leq \kappa \rangle, M_i <^1_{\lambda,\kappa} M_{i+1}, <_{\mathfrak{K}}$-increasing continuous $M_0 = M, M_\kappa = N$ and as in the proof of 6.3 without loss of generality $M_i = EM(\alpha_i, \Phi)$ where $\alpha_i < \mu^+$.

To prove $N = N_\kappa$ is μ-saturated suppose $p \in \mathscr{S}^1(M^*), M^* \leq_{\mathfrak{K}} N, \|M^*\| < \mu$; as we can extend M^* (as long as its power is $< \mu$ and it is $<_{\mathfrak{K}} N$), without loss of generality $M^* = EM(J, \Phi), J \subseteq \alpha_\kappa, |J| < \mu$.

So for some γ we have $[\gamma, \gamma + \omega) \cap J = \emptyset$ and $\gamma + \omega \leq \alpha_\kappa$. We can replace $[\gamma, \gamma + \omega)$ by a copy of λ; this will make the model μ-saturated [alternatively, use $I^* \times$ ordinal as in a previous proof].
But easily this introduces no new types realized over M^*. So p is realized.
2) Follows by part (1) or its proof.
3) Follows from the proof of part 1). Left to the reader.

Remark. In part (1) we have used just $\text{cf}(\lambda) > \mu > LS(\mathfrak{K})$. $\qquad \square_{6.5}$

6.7 Claim. *Assume K categorical in λ, $\mathrm{cf}(\lambda) > \mu > LS(\mathfrak{K})$. If $N_i \in K_\mu$ is saturated, increasing with i for $i < \delta$ and $\delta < \mu^+$ <u>then</u> $N = \bigcup_{i < \delta} N_i \in K_\mu$ is saturated.*

Proof. We prove this by induction on δ, so without loss of generality $\langle N_i : i < \delta \rangle$ is not just $\leq_{\mathfrak{K}}$-increasing and also contradicts the conclusion but is increasing continuous and each N_i saturated. Without loss of generality $\delta = \mathrm{cf}(\delta)$. If $\mathrm{cf}(\delta) = \mu$ the conclusion clearly holds so assume $\mathrm{cf}(\delta) < \mu$. Let $M \leq_{\mathfrak{K}} N, \|M\| < \mu$ and $p \in \mathscr{S}(M)$ be omitted in N and let $\theta = \delta + \|M\| + LS(\mathfrak{K}) < \mu$, and let $p \leq q \in \mathscr{S}(N)$. Now we can choose by induction on $i \leq \delta, M_i \leq N_i$ and $M_i^+ \leq_{\mathfrak{K}} N$ such that $M_i \in K_\theta, M_i^+ \in K_\theta, M_i$ is $\leq_{\mathfrak{K}}$-increasing continuous and $M \cap N_i \subseteq M_i, j < i \Rightarrow M_j^+ \cap N_i \subseteq M_{i+1}$ and $M_i <_{\theta,\omega}^1 M_{i+1}$ and if q does θ-split over M_i then $q \upharpoonright M_i^+$ does θ-split over M_i.

So by 6.3, 6.5 we know that M_δ is saturated, and for some $i(*) < \delta$ we have: $q \upharpoonright M_\delta$ does not θ-split over $M_{i(*)}$. But $M_{i(*)}^+ \subseteq N = \bigcup_{i < \delta} N_i, M_{i(*)}^+ \cap N_j \subseteq M_{j+1}$ so $M_{i(*)}^+ \subseteq M_\delta$. So necessarily $q \in \mathscr{S}(N)$ does not θ-split over $M_{i(*)}$.

Now we choose by induction on $\alpha < \theta^+, M_{i(*),\alpha}, b_\alpha, f_\alpha$ such that: $M_{i(*),\alpha} \in K_\theta, M_{i(*)} \leq_{\mathfrak{K}} M_{i(*),\alpha} \leq_{\mathfrak{K}} N_{i(*)}, M_{i(*),\alpha}$ is $\leq_{\mathfrak{K}}$-increasing continuous in $\alpha, b_\alpha \in N_{i(*)}$ realizes $q \upharpoonright M_{i(*),\alpha}, f_\alpha$ is a function with domain M_δ and range $\subseteq N_{i(*)}$ such that the sequences $\bar{c} = \langle c : c \in M_\delta \rangle$ and $\bar{c}^\alpha =: \langle f_\alpha(c) : c \in M_\delta \rangle$ realize the same type over $M_{i(*),\alpha}$ and $\{b_\alpha\} \cup \mathrm{Rang}(f_\alpha) \subseteq M_{i(*),\alpha+1}$. As $N_{i(*)}$ is saturated we can carry the construction; if some b_α realizes $q \upharpoonright M_\delta$ we are done (as $b_\alpha \in N$ realizes p). Let $d \in \mathfrak{C}$ realize q so

$(*)_1$ $\alpha < \beta < \theta^+ \Rightarrow \bar{c}^\beta\,{}^\frown\langle b_\alpha \rangle$ does not realize $\mathrm{tp}(\bar{c}\,{}^\frown\langle d \rangle, M_{i(*)}, \mathfrak{C})$.
      [Why? As $\bar{c}\,{}^\frown\langle b_\alpha \rangle$ does not realize $\mathrm{tp}(\bar{c}\,{}^\frown\langle d \rangle, M_{i(*)}, \mathfrak{C})$ because d realizes $p \upharpoonright \bar{c}$ whereas b_α does not realize $p \upharpoonright \bar{c}$.]

On the other hand as q does not θ-split over $M_{i(*)}$ we have $\mathrm{tp}(\bar{c}\,{}^\frown\langle d \rangle, M_{i(*)}, \mathfrak{C}) = \mathrm{tp}(\bar{c}^\alpha\,{}^\frown\langle d \rangle, M_{i(*)}, \mathfrak{C})$ so by the choice of b_β:

$(*)_2$ if $\alpha < \beta < \theta^+$ then $\bar{c}^\alpha\,{}^\frown\langle b_\beta \rangle$ realizes $\mathrm{tp}(\bar{c}\,{}^\frown\langle d \rangle, M_{i(*)}, \mathfrak{C})$.

We are almost done by 4.15.
[Why only almost? We would like to use the "θ-order property fail", now if we could define $\langle \bar{c}^\beta\,{}^\frown\langle b_\beta \rangle : $ for $\beta < (2^\theta)^+ \rangle$ fine, but we have only $\alpha < \theta^+$, this is too short.] Now we will refine the construction to make $\langle \bar{c}^\beta\,{}^\frown\langle b_\beta \rangle : \beta < \theta^+ \rangle$ strictly indiscernible which will be enough. As $N_{i(*)}$ is saturated without loss of generality $N_{i(*)} = EM_{\tau(\mathfrak{K})}(\mu, \Phi)$ and $M_{i(*)} = EM_{\tau(\mathfrak{K})}(\theta, \Phi)$ (using 6.8 below). As before for some $\gamma < \theta^+$ there are sequences $\bar{c}', \bar{b}'$ in $EM_{\tau(\mathfrak{K})}(\mu + \gamma, \Phi)$ realizing $\mathrm{tp}(\bar{c}, N_{i(*)}, \mathfrak{C}), q \upharpoonright N_{i(*)}$ respectively, here we use $\mathrm{cf}(\lambda) > \mu$ rather than just $\mathrm{cf}(\lambda) \geq \mu$. For each $\beta < \theta^+$ there is a canonical isomorphism g_β from $EM_{\tau(\Phi)}(\beta \cup [\mu, \mu + \gamma), \Phi)$ onto $EM_{\tau(\Phi)}(\beta + \gamma, \Phi)$. So without loss of generality $M_{i(*),\alpha} = EM_{\tau(\mathfrak{K})}(\theta + \gamma_\alpha, \Phi), \bar{c}^\alpha = g_{\theta+\gamma_\alpha}(\bar{c}'), b_\alpha = g_{\theta+\gamma_\alpha}(b')$. So $(*)_1 + (*)_2$ gives the order property. $\square_{6.7}$

We really proved, in 6.5 (from λ categoricity):

6.8 Subfact. Assume K is categorical in λ.
1) If $I \subseteq J$ are linear order, of power $< \text{cf}(\lambda)$;

$(*) \quad t \in J\backslash I \Rightarrow \left(\exists^{\aleph_0} s \in J \right)[s \sim_I t]$ where $s \sim_I t$ means "s, t realize the same Dedekind cut",

<u>then</u> every type over $EM_{\tau(\mathfrak{K})}(I, \Phi)$ is realized in $EM_{\tau(\mathfrak{K})}(J, \Phi)$.
2) Adding more Skolem functions we can omit $(*)$, for a suitable Φ we can make even the extension μ-saturated over $EM_{\tau(\mathfrak{K})}(I, \Phi)$.

Proof. Why? Use the proof of 6.5(1).
Replace the cut of t in I by λ: we get $\text{cf}(\lambda)$-saturated model. $\square_{6.8}$

§7 More on Splitting

7.1 Hypothesis. As before + conclusions of §6 for $\mu \in [LS(\mathfrak{K}), \mathrm{cf}(\lambda))$.
So

$(*)(a)$ $\mathfrak{K}$ has a saturated model in μ.

   (b) union of increasing chain of saturated models in K_μ of length $\leq \mu$
      is saturated.

   (c) <u>if</u> $\langle M_i : i \leq \delta \rangle$ increasing continuous in K_μ, each M_{i+1} saturated over M_i
      (the previous one), $p \in \mathscr{S}(M_\delta)$ <u>then</u> for some $i < \delta$, p does not μ-split over
      M_i.

7.2 Conclusion. If $p \in \mathscr{S}^m(M)$ and $M \in K_\mu$ is saturated, <u>then</u> for some
$M^- <^1_{\mu,\omega} M, M^- \in K_\mu$ is saturated and p does not μ-split over M^-.

Proof. We can find $\langle M_n : n \leq \omega \rangle$ in K_μ, each M_n saturated $M_n \leq^1_{\mu,\omega} M_{n+1}$ and
$M_\omega = \bigcup\limits_{n < \omega} M_n$ so as M_ω is saturated, without loss of generality $M_\omega = M$. Now
using $(*)(c)$ of 7.1 some M_n is O.K. as M^-. $\square_{7.2}$

7.3 Fact. If $M_0 \leq^1_{\mu,\omega} M_2 \leq^1_{\mu,\omega} M_3, p \in \mathscr{S}^m(M_3)$, p does not μ-split over M_0, <u>then</u>
$R(p) = R(p \restriction M_2)$.

Proof. We can find (by uniqueness) $M_1 \in K_\mu$ such that $M_0 \leq^1_{\mu,\omega} M_1 \leq^1_{\mu,\omega} M_2$ and
we can find $M_4 \in K_\mu$ such that $M_3 \leq^1_{\mu,\omega} M_4$.
We can find an isomorphism h_1 from M_3 onto M_2 over M_1 (by the uniqueness
properties $<^1_{\mu,\omega}$). By uniqueness there is an automorphism h of M_4 extending h_1.
Also by uniqueness there is $q \in \mathscr{S}(M_4)$ which does not μ-split over M_0 and extend
$p \restriction M_1$. As $p, p \restriction M_2$ does not μ-split over M_0 and have the same restriction to
M_1 and $M_0 \leq_{\mu,\omega} M_1$ clearly $p = q \restriction M_2$. Consider q and $h(q)$ both from $\mathscr{S}(M_3)$,
both do not μ-split over M_0 and have the same restriction to M_1; as $M_0 <^1_{\mu,\omega} M_1$
it follows that $q = h(q)$.
So $R(p \restriction M_1) = R(q \restriction M_1) = R(h(q \restriction M_2)) = R(q \restriction M_2) = R(p)$ as required.
$\square_{7.3}$

7.4 Claim. *[K categorical in λ, $\mathrm{cf}(\lambda) > \mu > LS(\mathfrak{K})$].*
   Suppose $m < \omega, M \in K_\mu$ *is saturated,* $p \in \mathscr{S}^m(M), M \leq_\mathfrak{K} N \in K_\mu, p \leq q \in$
$\mathscr{S}^m(N), N$ *saturated over* M, q *not a stationarization of* p *(i.e. for no* $M^- <^\circ_{\mu,\omega} M$,
q *does not μ-split over* M^- *).* <u>*Then*</u> q *does μ-divide over* M.

Proof. By 7.5 below and 6.3 (just p does not μ-split over some N_m where
$\langle N_\alpha : \alpha \leq \omega \rangle$ witness $N_0 <^1_{\mu,\omega} M$).

7.5 Claim. *[Assumptions of 7.5] Assume $M_0 <^1_{\mu,\omega} M_1 <^1_{\mu,\omega} M_2$ all saturated. If $q \in \mathscr{S}(M_2)$ does not μ-split over M_1 and $q \upharpoonright M_1$ does not μ-split over M_0, <u>then</u> q does not μ-split over M_0.*

Proof. Let $M_3 \in K_\mu$ be such that $M_2 <^1_{\mu,\omega} M_3$ and $c \in M_3$ realizes q. Choose a linear order I^* such that $I^* \times (\mu + \omega^*) \cong I^* \cong I^* \times \mu$, remember that on the product we do not use lexicographic order. I^* has no first nor last element (see [Sh 220, AP]).

Let $I_0 = I^* \times \mu, I_1 = I_0 + I^* \times \mathbb{Z}, I_2 = I_1 + I^* \times \mathbb{Z}, I_3 = I_2 + I^* \times \mu$. Clearly without loss of generality $M_\ell = EM_{\tau(\mathfrak{K})}(\Phi, I_\ell)$, let $c = \tau(\bar{a}_{t_0}, \ldots, a_{t_k})$ so $t_0, \ldots, t_k \in I_3$; let $I_{1,n} = I_0 + I^* \times \{m : \mathbb{Z} \models m < n\}$ and $I_{2,n} = I_0 + I^* \times \{m : \mathbb{Z} \models m < n\}$ and $I_{0,\alpha} = \alpha \times I^*$. So we can find a (negative) integer $n(*)$ small enough and $m(*) \in \mathbb{Z}$ large enough such that $\{t_0, \ldots, t_n\} \cap I_{2,n(*)+1} \subseteq I_{1,m(*)-1}$. Let $M_{1,n} = EM_{\tau(\mathfrak{K})}(I_{1,n}, \Phi)$ and $M_{2,n} = EM_{\tau(\mathfrak{K})}(I_{2,n}, \Phi)$. Clearly $M_0 <^1_{\mu,\omega} M_{1,n} <^1_{\mu,\omega} M_1 <^1_{\mu,\omega} M_{2,n} <^1_{\mu,\omega} M_2$. Clearly (use automorphism of I_3)

$(*)_0$ $q \upharpoonright M_{2,n}$ does not μ-split over $M_{1,m}$ if $\mathbb{Z} \models n < n(*), m(*) \leq m \in \mathbb{Z}$.

By 7.3 with $q, M_1, M_{2,n}, M_2, q$ here standing for M_0, M_2, M_3, p there we get

$(*)_1$ $R(q) = R(q \upharpoonright M_{2,n})$ if $n \in \mathbb{Z}$.

Similarly

$(*)_2$ $R(q \upharpoonright M_1) = R(q \upharpoonright M_{1,m})$ if $m \in \mathbb{Z}$.

By $(*)_0$ and 7.3 we have

$(*)_3$ $R(q \upharpoonright M_{2,n(*)}) = R(q \upharpoonright M_{1,m(*)})$.

Similarly we can find a successor ordinal $\alpha(*) < \mu$ and $k(*) \in \mathbb{Z}$ such that

$$\{t_0, \ldots, t_k\} \cap I_{1,k(*)+1} \subseteq I_{0,\alpha(*)-1}$$

and then prove

$(*)_4$ $R(q \upharpoonright M_0) = R(q \upharpoonright M_{0,\alpha})$ if $\alpha(*) \leq \alpha < \mu$

$(*)_5$ $R(q \upharpoonright M_{1,\ell(*)}) = R(q \upharpoonright M_{0,\alpha})$ if $\alpha(*) \leq \alpha < \mu$.

Together $R(q) = R(q \upharpoonright M_0)$, hence q does not μ-split over M_0 as required. $\square_{7.5}$

PART II

§8 Existence[2] of nice Φ

We build EM models, where "equality of types over A in the sense of the existence of automorphisms over A" behaves nicely.

8.1 Context.

(a) $\mathfrak{K}$ is an abstract elementary class with models of cardinality $\geq \beth_{(2^{LS(\mathfrak{K})})^+}$; it really suffices to assume:

(a)$'$ $\mathfrak{K}$ is a class of $\tau(K)$-models, which is $PC_{\kappa^+,\omega}$ with a model of cardinality $\geq \beth_{(2^{LS(\mathfrak{K})})^+}$.

8.2 Definition. 1) Let $\kappa \geq LS(\mathfrak{K})$, now $\Upsilon_{\kappa}^{or} = \Upsilon_{\kappa,\tau}^{or}$ is the family of Φ proper for linear orders (see [Sh:c, Ch.VII]) such that:

(a) $|\tau(\Phi)| \leq \kappa$

(b) $EM_{\tau(\mathfrak{K})}(I, \Phi) = EM(I, \Phi) \restriction \tau(K) \in K$

(c) $I \subseteq J \Rightarrow EM_{\tau(\mathfrak{K})}(I, \Phi) \leq_{\mathfrak{K}} EM_{\tau(\mathfrak{K})}(J, \Phi)$.

2) Υ^{or} is $\Upsilon_{LS(\mathfrak{K})}^{or}$.

8.3 Definition. We define partial orders $\leq_{\kappa}^{\oplus}$ and $\leq_{\kappa}^{\otimes}$ on Υ_{κ}^{or} (for $\kappa \geq LS(\mathfrak{K})$):
1) $\Psi_1 \leq_{\kappa}^{\oplus} \Psi_2$ <u>if</u> $\tau(\Psi_1) \subseteq \tau(\Psi_2)$ and $EM_{\tau(\mathfrak{K})}(I, \Psi_1) \leq_{\mathfrak{K}} EM_{\tau(\mathfrak{K})}(I, \Psi_2)$ and $EM(I, \Psi_1) \subseteq EM_{\tau(\Psi_1)}(I, \Psi_2)$ and $EM(I, \Psi_1) = EM_{\tau(\Psi_1)}(I, \Psi_1) \subseteq EM_{\tau(\Psi_1)}(I, \Psi_2)$ for any linear order I.
Again for $\kappa = LS(\mathfrak{K})$ we may drop the κ.
2) For $\Phi_1, \Phi_2 \in \Upsilon_{\kappa}^{or}$, we say Φ_2 is an inessential extension of Φ_1, $\Phi_1 \leq_{\kappa}^{ie} \Phi_2$ if $\Phi_1 \leq_{\kappa}^{\oplus} \Phi_2$ and for every linear order I, we have

$$EM_{\tau(\mathfrak{K})}(I, \Phi_1) = EM_{\tau(\mathfrak{K})}(I, \Phi_2).$$

(note: there may be more functions in $\tau(\Phi_2)$!)

3) $\Phi_1 \leq_{\kappa}^{\otimes} \Phi_2$ <u>iff</u> there is Ψ proper for linear order and producing linear orders such that:

(a) $\tau(\Psi)$ has cardinality $\leq \kappa$,

(b) $EM(I, \Psi)$ is a linear order which is an extension of I: in fact $[t \in I \Rightarrow x_t = t]$

(c) $\Phi_2' \leq_{\kappa}^{ie} \Phi_2$ where $\Phi_2' = \Psi \circ \Phi_1$, i.e.

$$EM(I, \Phi_2') = EM(EM(I, \Psi), \Phi_1).$$

(So we allow further expansion by functions definable from earlier ones (composition or even definition by cases), as long as the number is $\leq \kappa$).

---

[2]Done end of Oct.1988

8.4 Claim. *1)* $(\Upsilon_\kappa^{or}, \leq_\kappa^\otimes)$ *and* $(\Upsilon_\kappa^{or}, \leq^\oplus)$ *are partial orders (and* $\leq_\kappa^\otimes \subseteq \leq_\kappa^\oplus$*).*
2) Moreover, if $\langle \Phi_i : i < \delta \rangle$ *is a* $\leq_\kappa^\otimes$*-increasing sequence,* $\delta < \kappa^+$, <u>*then*</u> *it has a*
$<_\kappa^\otimes$*-l.u.b.* Φ*;* $EM^1(I, \Phi) = \bigcup_{i < \delta} EM^1(I, \Phi_i)$.

3) Similarly for $<_\kappa^\oplus$.

8.5 Lemma. *1) If* $N \leq_\mathfrak{K} M, \|M\| \geq \beth_{(2^\chi)^+}, \chi \geq \|N\| + LS(\mathfrak{K})$, <u>*then*</u> *there is* Φ
proper for linear order such that:

(a) $EM_{\tau(\mathfrak{K})}(\emptyset, \Phi) = N$

(b) $N \leq_\mathfrak{K} EM_{\tau(\mathfrak{K})}(I, \Phi)$, *moreover*
$I \subseteq J \Rightarrow EM_{\tau(\mathfrak{K})}(I, \Phi) \leq_\mathfrak{K} EM_{\tau(\mathfrak{K})}(I, \Phi)$

(c) $EM_{\tau(\mathfrak{K})}(I, \Phi)$ *omits every type* $p \in \mathscr{S}(N)$ *which* M *omits, moreover if* I *is finite then* $EM_{\tau(\mathfrak{K})}(I, \Phi)$ *can be* $\leq_\mathfrak{K}$*-embedded into* M.

Proof. Straight by [Sh 88, 1.7] or deduce by 4.6 or use 8.6 with $N_1 = N_0$.

8.6 Lemma. *Assume*

(a) $LS(\mathfrak{K}) \leq \chi \leq \lambda$

(b) $N_0 \leq_\mathfrak{K} N_1 \leq_\mathfrak{K} M$

(c) $\|N_0\| \leq \chi, \|N_1\| = \lambda$ *and* $\|M\| \geq \beth_{(2^\chi)^+}(\lambda)$

(d) $\Gamma_0 = \{p_i^0 : i < i_0^*\} \subseteq \mathscr{S}(N_0)$ *each* p_i^0 *omitted by* M

(e) $\Gamma_1 = \{p_i^1 : i < i_1^* \leq \chi\} \subseteq \mathscr{S}(N_1)$ *such that for no* $i < i_1^*$ *any* $c \in M$ *does* c *realizes* p_i'/E_χ *[i.e. realizes each* $p_i' \upharpoonright M, M \leq_\mathfrak{K} N_1, M \in \mathfrak{K}_{\leq \chi}$*].*

<u>*Then*</u> *we can find* $\langle N_\alpha' : \alpha \leq \omega \rangle, \Phi$ *and* $\langle q_i^1 : i < i_1^* \rangle$ *such that*

(α) Φ *proper for linear order*

(β) $N_\alpha' \in \mathfrak{K}_{\leq \chi}$ *is* $\leq_\mathfrak{K}$*- increasing continuous (for* $\alpha \leq \omega$*)*

(γ) $N_0' = N_0$ *and* $N_\alpha' \leq_\mathfrak{K} N_1$

(δ) $q_i^1 \in \mathscr{S}(N_\omega')$

(ϵ) $EM_{\tau(\mathfrak{K})}(\emptyset, \Phi)$ *is* N_ω'

(ζ) *for linear order* $I \subseteq J$ *we have*
$EM_{\tau(\mathfrak{K})}(I, \Phi) \leq_\mathfrak{K} EM_{\tau(\mathfrak{K})}(J, \Phi)$

(η) *for each* n*, there is a* $\leq_\mathfrak{K}$*-increasing sequence* $\langle N_{n,m} : m < \omega \rangle$ *with union* $EM_{\tau(\mathfrak{K})}(n, \Phi)$ *and a* $\leq_\mathfrak{K}$*-embedding* $f_{n,m}$ *of* $N_{n,m}$ *into* M *with range* $N_{n,m}'$ *such that*

   (i) $N_m' = N_{0,m}'$,

   (ii) $f_{n,m} \upharpoonright N_0$ *is the identity,* $Rang(f_{0,m}) \subseteq N_1$

   (iii) $f_{n,m}(q_i^1 \upharpoonright N_m') = p_i^1 \upharpoonright Rang(f_n)$ *for* $i < i_1^*$

(θ) $EM_{\tau(\mathfrak{K})}(I, \Phi)$ *omits every* p_i^0 *for* $i < i_0^*$ *and omits every* q_i^1 *in a strong sense:* *for every* $a \in EM_{\tau(\mathfrak{K})}(I, \Phi)$ *for some* n *we have*
$q_i^1 \upharpoonright N_n' \neq tp(a, N_n', EM_{\tau(\mathfrak{K})}(I, \Phi))$.

Remark. 1) So we really can replace q_i^1 by $\langle q_i^1 \upharpoonright N'_n : n < \omega \rangle$, but for ω-chains by chasing arrows such limit (q_i^1) exists.

2) Clause (ζ) follows from Clause (η).

Proof. By [Sh 88, 1.7] (and see 0.5) we can find $\tau_1, \tau(\mathfrak{K}) \subseteq \tau_1, |\tau_1| \leq \chi$ (here we can have $|\tau_1| \leq LS(\mathfrak{K}) \leq \chi$) and an expansion M^+ of M to a τ_1-model and a set Γ of quantifier free types (so $|\Gamma| \leq 2^{\aleph_0 + |LS(\mathfrak{K})|}$) such that:

- (A) M^+ omits every $p \in \Gamma$ and if M^* is a τ_1-model omitting every $p \in \Gamma$ <u>then</u> $M^* \upharpoonright \tau(\mathfrak{K}) \in K$ and $N^* \subseteq M^* \Rightarrow N^* \upharpoonright \tau(\mathfrak{K}) \leq_{\mathfrak{K}} M^* \upharpoonright \tau(\mathfrak{K})$

- (B) for $\bar{a} \in {}^{\omega >}M$ we let $M_{\bar{a}}^+ = M^+ \upharpoonright c\ell(\bar{a}, M^+)$ then $M_{\bar{a}}^+ \upharpoonright \tau(\mathfrak{K}) \leq_{\mathfrak{K}} M^+ \upharpoonright \tau(\mathfrak{K})$, $\text{Rang}(\bar{a}) \subseteq \text{Rang}(\bar{b}) \Rightarrow M_{\bar{a}}^+ \upharpoonright \tau(\mathfrak{K}) \leq_{\mathfrak{K}} M_{\bar{b}}^+ \upharpoonright \tau(\mathfrak{K})$ where $\bar{a} \in {}^{\omega >}(N_\ell) \Rightarrow |M_{\bar{a}}^+| \subseteq N_\ell$.

Note: Further expansion of M^+ to M^*, as long as $|\tau(M^*)| \leq \chi$ preserves $(A) + (B)$ so we can add

- (C) $N_0, M_{\langle \rangle}^+$ have the same universe

  and let for $\ell = 0, 1$, $M_{\bar{a},\ell}^+ = M_{\bar{a}}^+ \upharpoonright (|N_\ell| \cap |M_{\bar{a}}^+|), \ell = 1, 2$

- (D) $M_{\bar{a},0}^+ \upharpoonright \tau(\mathfrak{K}) \leq_{\mathfrak{K}} M_{\bar{a},1}^+ \upharpoonright \tau(\mathfrak{K}) \leq_{\mathfrak{K}} M_{\bar{a}}^+ \upharpoonright \tau(\mathfrak{K})$

- (E) for $i < i_1^*$, the type $p_i^1 \upharpoonright (M_{\bar{a},1}^+ \upharpoonright \tau(\mathfrak{K}))$ is not realized in $M_{\bar{a}}^+ \upharpoonright \tau(\mathfrak{K})$.

Now we choose by induction on n, sequence $\langle f_\alpha^n : \alpha < (2^\chi)^+ \rangle$ and N'_n such that:

- (i) f_α^n is a one-to-one function from $\beth_\alpha(\lambda)$ into M

- (ii) $\langle f_\alpha^n(\zeta) : \zeta < \beth_\alpha(\lambda) \rangle$ is n-indiscernible in M^+

- (iii) moreover, if $\alpha, \beta < (2^\chi)^+$, and $m \leq n$ and $\zeta_1 < \ldots < \zeta_m < \beth_\alpha(\lambda)$ and $\xi_1 < \ldots < \xi_m < \beth_\beta(\lambda)$ <u>then</u>: the sequences $\bar{a} = \langle f_\alpha^n(\zeta_1), \ldots, f_\alpha^n(\zeta_m) \rangle$, $\bar{b} = \langle f_\beta^m(\xi_1), \ldots, f_\beta^m(\xi_m) \rangle$ realize the same quantifier free type in M^+ over N_1^+, so there is a natural isomorphism $g_{\bar{b},\bar{a}}$ from $M_{\bar{a}}^+$ onto $M_{\bar{b}}^+$ (mapping $f_\alpha(\zeta_\ell)$ to $f_\beta(\xi_1)$), moreover

$$i < i_1^* \Rightarrow g_{\bar{b},\bar{a}}(p_i^1 \upharpoonright (M_{\bar{a},1}^+ \upharpoonright \tau(\mathfrak{K}))) = p_i^1 \upharpoonright (M_{\bar{b},1}^+ \upharpoonright \tau(\mathfrak{K}))$$

and

$$N'_m = M_{\bar{a},1}^+ \upharpoonright \tau(\mathfrak{K}).$$

The rest should be clear. $\square_{8.6}$

8.7 Claim. *Suppose*

- (a) $\Phi \in \Upsilon_\kappa^{or}$

- (b) $n < \omega$, u, u_1, u_2 *are subsets of* $\{0, 1, \ldots, n-1\}$ *and* $\sigma_1(\ldots, \bar{x}_\ell, \ldots)_{\ell \in u_1}$, $\sigma_2(\ldots, \bar{x}_\ell, \ldots)_{\ell \in u_2}$ *are* $\tau(\Phi)$-*terms.*

- (c) *for every* $\alpha < (2^{LS(\mathfrak{K})})^+$ *(or at least* $\beth_\alpha < \mu(\kappa)$ *- see [Sh:c, Ch.VII,§4] but for this we should be careful as to omit only* $\leq LS(\mathfrak{K})$ *types) there are linear*

orders $I \subseteq J, I \ \aleph_0$-homogeneous inside[3] J, I of cardinality $\geq \beth_\alpha$, such that
for some (equivalently every) $t_0 < t_1 < \cdots < t_{n-1}$ of I we have:

$\oplus$ for some automorphism f of $EM_{\tau(\mathfrak{K})}(J, \Phi)$,
   $f \restriction EM_\tau(I \backslash \{t_\ell : \ell < n, \ell \notin u\}, \Phi)$ is the identity and
$$f\left(\sigma_1(..., \bar{a}_{t_\ell}, ...)_{\ell \in u_1}\right) = \sigma_2(..., \bar{a}_{t_\ell}, ...)_{\ell \in u_2}.$$

THEN for some Φ', $\Phi \leq_\kappa^\oplus \Phi'$ and even $\Phi \leq_\kappa^\otimes \Phi'$ we have

$\otimes$ for every linear order I and $t_0 < \cdots < t_{n-1}$ from I, there is an automor-
   phism f of $EM_\tau(I, \Phi')$ such that:

(α) $f \restriction EM(I \backslash \{t_\ell : \ell < n, \ell \notin u\}, \Phi')$ is the identity and

(β) $f\left(\sigma_1(..., \bar{a}_{t_\ell}, ...)_{\ell \in u_1}\right) = \sigma_2(..., \bar{a}_{t_\ell}, ...)_{\ell \in u_2}$

(γ) $f = F(-, \bar{a}_{t_0}, \ldots, \bar{a}_{t_{n-1}})$ for some $F \in \tau(\Phi')$.

Proof. Expand $M = EM(J, \Phi)$ by the predicates $Q_1 = \{\bar{a}_t : t \in I\}, Q_2 = \{\bar{a}_t : t \in J\}$ if $\alpha = \ell g(\bar{a}_t)$ is finite, in any case we use $Q_{\ell,i,j} = \{(a_{t,i}, a_{t,j}) : t \in I\}$ for $\ell \in \{1, 2\}$ and $i \leq j < \alpha$; and without loss of generality $t \neq s \Rightarrow a_{t,0} \neq a_{s,0}$ and we identify $t \in J$ with $a_{t,0}$. For $t_0 < \ldots < t_{n-1} \in I$, let $f_{t_0,\ldots,t_{n-1}} \in AUT(EM_{\tau(\mathfrak{K})}(J, \Phi))$ be as in ($\oplus$) and let g_ℓ (for $\ell < \omega$) be functions from M into $\{\bar{a}_t : t \in J\}$ such that $\forall x \in M, x = \sigma_x(g_0(x) \ldots g_{n-1}(x))$ such that $g_\ell(x) <^J g_{\ell+1}(x)$ if $\ell < n - 1$ and $g_\ell(x) = g_{\ell+1}(x)$ otherwise. Lastly, let $P_\sigma = \{x : \sigma_x = \sigma\}$.

Let F be an $(n+1)$-ary function, $F(\bar{a}_{t_0}, \ldots, \bar{a}_{t_{n-1}}, b) = f_{t_0 \ldots t_{n-1}}(b)$ when defined. The model we get we call M^+. Now use the omitting types theorem, e.g. 8.5. So there is a model N^+ and $\langle \bar{b}_n : n < \omega \rangle$ indiscernible in it such that $N^+ \equiv M^+, N^+$ omits all types which M^+ omits, for every $m < \omega$ for some $s_0 < \ldots < s_{n-1}$ from I the type of $\bar{b}_0 \hat{\ } \ldots \bar{b}_{n-1}$ in N^+ is equal to the type of $\bar{a}_{s_0} \hat{\ } \ldots \hat{\ } \bar{a}_{s_{n-1}}$ in M^+. Define Φ' such that $EM(I^*, \Phi')$ is a $\tau(N^+)$-model generated by $\{\bar{a}_t : t \in I^*\}$ such that $t_0 < \ldots < t_{n-1} \in I^* \Rightarrow$ type of $\bar{a}_{t_0} \hat{\ } \ldots \hat{\ } \bar{a}_{t_{n-1}}$ in $EM(I^*, \Phi')$ is equal to type of $\bar{b}_0 \hat{\ } \ldots \hat{\ } \bar{b}_{n-1}$ in N^+.

Why is $\Phi <^\otimes \Phi'$ and not just $\Phi <^\oplus \Phi'$?
Here we use[4] Q_1, Q_2 in M^+ we have

(∗)   every $c \in M^+$ is in the τ_Φ-Skolem Hull of $Q_2^{M^+} = \{\bar{a}_t : t \in J\}$.

So

(∗)′ M^+ omits the type

$$p(x) = \left\{ \neg(\exists \bar{y}_0, \ldots, \bar{y}_{n-1})\left(\bigwedge_{\ell < n} Q_2(\bar{y}_\ell) \ \& \ x = \sigma(y_0, \ldots, y_n) : \sigma \in \tau_\Phi\right) \right\}.$$

$\square_{8.7}$

---

[3]this means that every partial order preserving function h from I to I can be extended to an automorphism of J.
[4]if $\ell g(\bar{a}_t)$ is infinite, slightly more complicated

8.8 Conclusion. For $\kappa \geq LS(\mathfrak{K})$ there is $\Phi^* \in \Upsilon_\kappa^{or}$ (in fact for every $\Phi \in \Upsilon_\kappa^{or}$ there is $\Phi^*, \Phi \leq_\kappa^\otimes \Phi^* \in \Upsilon_\kappa^{or}$) satisfying:

- (a) if Φ^* satisfies the assumptions of 8.7 for some I, J (playing the role of Φ there) <u>then</u> it satisfies its conclusion (i.e. playing the role of Φ' there)

- (b) moreover if $\kappa \geq 2^{LS(\mathfrak{K})}$, for some $\chi(\Phi^*) < \mu(\kappa)$ (see [Sh:c, Ch.VII,§4]), we can weaken the assumption $\alpha < \beth_{(2^\kappa)^+}$ to $\beth_\alpha \leq \chi(\Phi^*)$

- (c) moreover, in 8.7 we can omit "I is $\aleph_0$-homogeneous inside J"

- (d) also we can replace clause (α) of $\otimes$ (of 8.7) by: f extends some automorphism of $EM(I\backslash\{t_\ell : \ell < n, \ell \notin u\}, \Phi^*)$ definable as in clause (γ) of $\otimes$ of 8.7.

- (e) we can deal similarly with automorphisms extending a given $f \restriction EM(I \restriction \{t_\ell : \ell < n\})$ and having finitely many demands.

Proof. For (a) iterate 8.7, by bookkeeping looking at all $\langle \sigma_1, \sigma_2, u, u_1, u_2 \rangle$ and use 8.4 for noting that the iteration is possible. Now (b) holds as $\mathrm{cf}(\mu(\kappa)) > \kappa$, and the number of terms is $\leq \kappa$. For (c) we can let Ψ be such that $EM(I, \Psi)$ is an $\aleph_0$-homogeneous linear order, $|\tau(\Psi)| = \aleph_0$ and use $\Psi \circ \Phi^*$. The rest are easy, too.
$$\square_{8.8}$$

8.9 Lemma. *Let Φ^* be as in 8.8, and I be a linear order of cardinality $\chi(\Phi^*)$ (where $\chi(\Phi^*)$ is from 8.8). Assume $\sigma(\bar{x}_0, \ldots, \bar{x}_{n-1})$ is a term in $\tau(\Phi^*)$, for $\ell = 1, 2$ we have $t_0^\ell < \cdots < t_{n-1}^\ell$ and $u \subseteq \{\ell : t_\ell^1 = t_\ell^2\}$, and there is no automorphism f of $EM(I, \Phi^*)$ such that $f \restriction EM(I\backslash\{t_i^1, t_i^2 : \ell < n, \ell \notin u\}, \Phi^*)$ is the identity, and $f\big(\sigma\big(\bar{a}_{t_0^1}, \ldots, \big)\big) = \sigma\big(\bar{a}_{t_0^2}, \ldots, \big)$.*
<u>*Then*</u>

- (1) *for $\chi > \chi(\Phi^*)^+$ we have $I(\chi, K) = 2^\chi$.*
- (2) *We have the $\chi(\Phi^*)$-order property in the sense of Definition 4.3 (see more [Sh 300, Ch.III,§3] or better [Sh:e, Ch.III,§3].)*

Proof. Without loss of generality I is dense.
    We can find $t_0^3 < \ldots < t_{n-1}^3$ such that
$$\ell \in u \Rightarrow t_\ell^3 = t_\ell^1,$$

$$\ell \notin u \Rightarrow t_\ell^3 \notin \{t_m^1, t_m^2 : m < n\}.$$

Now

- $\otimes_1$ there is no automorphism f of $EM(I, \Phi^*)$ such that
$f \restriction EM_\tau(I\backslash\{t_\ell^1, t_\ell^2, t_\ell^3 : \ell < n, \ell \notin u\}, \Phi)$ is the identity and
$f(\sigma(\bar{a}_{t_0^1}, \ldots)) = \sigma(\bar{a}_{t_0^2}, \ldots)$
[Why? If there is, easily some Φ contradicts 8.8(a)]

- $\otimes_2$ for some $k \in \{1, 2\}$, there is no automorphism f of $EM(I, \Phi)$ which is the identity of $EM(I\backslash\{t_\ell^1, t_\ell^2, t_\ell^3 : \ell < n, \ell \notin u\}, \Phi)$ and $f(\sigma(\bar{a}_{t_0^k}, \ldots)) =$

$\sigma(\bar{a}_{t_0^3}, \dots)$

[Why? If not such f_1, f_2 exists and $f_2^{-1} \circ f_1$ contradict $(*)_2$].

$\bigotimes_3$ for some $k \in \{1, 2\}$ there is no automorphism f of $EM(I, \Phi)$ which is the identity on $EM(I \backslash \{t_\ell^k, t_\ell^3 : \ell < n, \ell \notin u\}, \Phi)$ and $f(\sigma(\bar{a}_{t_0^k}, \dots)) = \sigma(a_{t_0^3}, \dots)$
[Why? We negate a stronger demand than in $(*)_2$].

By renaming we get that without loss of generality

$$t_\ell^1 = t_k^2 \Rightarrow \ell = k \in u.$$

By the transitivity of "there is an automorphism" we can assume that just for a singleton $\ell(*), t_{\ell(*)}^1 \neq t_{\ell(*)}^2$. Now if we increase u, surely such isomorphism does not exist so without loss of generality $u = \{\ell < n : \ell \neq \ell(*)\}$ and $t_{\ell(*)}^1 <_I t_{\ell(*)}^2$, by symmetry. Let $I^0 = \{t \in I : t <_I t_{\ell(*)}^1\}, I^1 = \{t \in I : t_{\ell(*)}^1 \leq_I t <_I t_{\ell(*)}^2\}, I^2 = \{t \in I : t_{\ell(*)}^2 <_I t\}$ (yes: $<_I$ not $\leq_I$).

Now for every linear order J we can define $I(J)$ as follows: $I(J)$ is a linear order which is the sum $I^0 + \sum_{t \in J} I_t^1(J) + I^2$, $I_t^1(J)$ is isomorphic to I^1, so let $f_t : I^1 \to I_t^1(J)$ be such an isomorphism. Let $\bar{b}^t$ list $EM(I^0 + I_t^1(J) + I^2)$ (such that for $t, s, (\mathrm{id}_{I^0} + f_s f_t^{-1} + \mathrm{id}_{I^2})$ induces a mapping from $\bar{b}^t$ onto $\bar{b}^s$). Let $\bar{c}^t = f_t(\sigma(t_0^1, \dots, t_{n-1}^1))$. Now

$(*)_1$ if $s_0 <_J r <_J s_1$ then there is no automorphism f of $EM_\tau(I(J), \Phi^*)$ over $\bar{b}^r$ mapping $\bar{c}^{s_0}$ to $\bar{c}^{s_1}$,

$(*)_2$ if J is $\aleph_0$-homogeneous (or just 2-transitive) and $r <_J s_0$ & $r <_J s_1$ or $s_0 <_J r$ & $s_1 <_J r$ then there is an automorphism f of $EM_\tau(I(J), \Phi^*)$ over $\bar{b}^r$ mapping $\bar{c}^{s_0}$ to $\bar{c}^{s_1}$.

So by [Sh:e, Ch.III,§3] (or earlier version [Sh 300, Ch.III,§3]), we have the order property for sequences of length $\chi(\Phi^*)$; the formula appearing in the definition of the order is preserved by automorphisms of the model; though it looks as second order, it does not matter. So conclusion (2) holds and (1) follows. $\square_{8.9}$

8.10 Claim. *Assume*

(a) *K is categorical in λ*

(b) *the $M \in K_\lambda$ is χ^+-saturated (holds if $cf(\lambda) > \chi$)*

(c) *$\chi \geq LS(\mathfrak{K})$.*

<u>*Then*</u> *every $M \in K$ of cardinality $\geq \beth_{(2^\chi)^+}$ (or just $\geq \beth_{\mu(\chi)}$ if $\chi \geq 2^{LS(\mathfrak{K})}$) is χ^+-saturated.*

Proof. If M is a counterexample, let $N \leq_\mathfrak{K} M, \|N\| \leq \chi$ and $p \in \mathscr{S}(N)$ be omitted by N. By the omitting type theorem for abstract elementary classes (see 8.5, i.e. [Sh 88]), we get $M' \in K_\lambda, N \leq_K M', M'$ omitting p a contradiction. $\square_{1.9}$

8.11 Claim. *Assume*

   (a) *$LS(\mathfrak{K}) \leq \chi$*

   (b) *for every $\alpha < (2^\chi)^+$ there are $M_\alpha <_{\mathfrak{K}} N_\alpha$ (so $M_\alpha \neq N_\alpha$), $\|M_\alpha\| \geq \beth_\alpha$ and
$p \in \mathscr{S}(M_\alpha)$ such that $c \in N_\alpha \Rightarrow \neg p E_\chi \ tp(c, M_\alpha, \mathfrak{C})$.*

1) For every $\theta > \chi$ there are $M <_{\mathfrak{K}} N$ in K_θ and $p \in \mathscr{S}(M_\alpha)$ as in clause (b).
*2) Moreover, if Φ is proper for orders as usual, $|\tau(\Phi)| \leq \chi, \beth_{(2^\chi)^+} \leq \lambda, K$ categorical
in λ and I a linear order of cardinality θ, <u>then</u> we can demand $M = EM_{\tau(\mathfrak{K})}(I, \Phi)$.*

Proof. Straight.

§9 Small Pieces are Enough and Categoricity

9.1 Context.

   (i) $\mathfrak{K}$ an abstract elementary class

  (ii) K categorical in $\lambda, \lambda > LS(\mathfrak{K})$

 (iii) some ($\equiv$ any) $M \in K_\lambda$ is saturated (if λ is regular this holds)

 (iv) Φ^* is as in 8.8.

Hence

9.2 Fact. For $\mu \in [LS(\mathfrak{K}), \lambda)$, there is a saturated model of cardinality μ,
(why? by 6.5(3)) and there is also $\Phi^* \in \Upsilon_\mu^{or}$ as in 8.8.

9.3 Main Claim. *If $M \in K$ is a saturated model of cardinality χ,
$\chi(\Phi^*) < \chi <$ cf$(\lambda) \leq \lambda$ <u>then</u> $\mathscr{S}(M)$ has character $\leq \chi(\Phi^*)$, i.e. if $p_1 \neq p_2$ are in
$\mathscr{S}(M)$ then for some $N \leq_\mathfrak{K} M, N \in K_{\chi(\Phi^*)}$ we have $p_1 \restriction N \neq p_2 \restriction N$.*

Proof. We can find $I \subseteq J, |I| = \chi, |J| = \lambda, M = EM_{\tau(\mathfrak{K})}(I, \Phi^*) \leq_\mathfrak{K} N^* = EM_{\tau(\mathfrak{K})}(J, \Phi^*)$ and I, J are $\aleph_0$-homogeneous. So by 6.7: every $p \in \mathscr{S}(M)$ is realized in N^* and say p is realized by $\sigma_p(\bar{a}_{t_{p,0}}, \bar{a}_{t_{p,1}}, \ldots, \bar{a}_{t_{p,n_p-1}})$ where $t_{p,0} < t_{p,1} < \ldots < t_{p,n_p-1}$. If the conclusion fails, then we can find $q \neq p$ in $\mathscr{S}(M)$ such that

   (∗) $N \leq_\mathfrak{K} M, \|N\| \leq \chi(\Phi^*) \Rightarrow p \restriction N = q \restriction N$.

Choose $I' \subseteq J, |I'| = \chi(\Phi^*)$ such that $I \subseteq I'$ and $\{t_{p,\ell} : \ell < n_p\} \subseteq I'$ and
$\{t_{q,\ell} : \ell < n_q\} \subseteq I'$ and let $M' = EM_{\tau(\mathfrak{K})}(I' \cap I, \Phi^*) \leq_\mathfrak{K} M$.
   So $p \restriction M' = q \restriction M'$; so 8.7 becomes relevant (i.e. 8.8(b)) considering the
$\aleph_0$-homogeneity of J) hence by the choice of $\Phi^*, p = q$ contradiction. $\square_{9.3}$

9.4 Conclusion. Let I be a directed partial order. Assume $M \in K_\chi$ is saturated,
$\chi(\Phi^*) \leq \chi < \lambda, \langle M_t : t \in I \rangle$ is a $\leq_\mathfrak{K}$-increasing family of $\leq_\mathfrak{K}$-submodels of M, each
saturated and $[t < s \Rightarrow M_t$ saturated over $M_s]$ and $\|M_t\| \leq \chi(\Phi^*)$, <u>then</u> for every

$$p \in \mathscr{S}\left(\bigcup_{t \in I} M_t\right) \text{ for some } t^* \in I:$$

   (∗) p does not μ-split over M_{t^*}
     (and even does not χ-split over M_{t^*}).

Proof. Clear by the proof of 9.3.

9.5 Claim. *If T is categorical in $\lambda, LS(\mathfrak{K}) \leq \chi(\Phi^*) \leq \mu < \lambda$ and $\langle M_i : i < \delta \rangle$ an $<_\mathfrak{K}$-increasing sequence of μ^+-saturated models <u>then</u> $\bigcup_{i<\delta} M_i$ is μ^+-saturated.*

Remark. 1) Hence this holds for limit cardinals $> LS(\mathfrak{K})$.
2) The addition compared to 6.7 is the case $\mathrm{cf}(\lambda) = \mu^+$, e.g. $\lambda = \mu^+$.

Proof. Let $M_\delta = \bigcup_{i<\delta} M_i$ and assume M_δ is not μ^+-saturated. So there are $N \leq_\mathfrak{K} M_\delta$ of cardinality $\leq \mu$ and $p \in \mathscr{S}(N)$ which is not realized in M_δ. Choose $p_1 \in \mathscr{S}(M_\delta)$ such that $p_1 \restriction N = p$.
Without loss of generality N is saturated.
   Let $\chi = \chi(\Phi^*)$, without loss of generality $\delta = \mathrm{cf}(\delta)$.
We claim

> $\otimes$ there are $i(*) < \delta$ and $N^* \leq_\mathfrak{K} M_{i(*)}$ of cardinality χ such that p does not χ-split over N^*.
> Why? Assume toward contradiction that this fails. The proof of $\otimes$ splits to two cases.

<u>Case I</u>: $\mathrm{cf}(\delta) \leq \chi$.
   We can choose by induction on $i < \delta, N_i^0, N_i^1$ such that

   (a) $N_i^0 \leq_\mathfrak{K} M_i$ has cardinality χ
   (b) N_i^0 is increasingly continuous
   (c) $N_i^0 <_{\chi,\omega}^1 N_{i+1}^0$
   (d) $N_i^0 \leq_\mathfrak{K} N_i^1 \leq_\mathfrak{K} M_\delta$
   (e) N_i^1 has cardinality $\leq \chi$
   (f) $p_1 \restriction N_i^1$ does χ-split over N_i^0
   (g) for $\varepsilon, \zeta < i, N_\varepsilon^1 \cap M_\zeta \subseteq N_i^0$.

There is no problem to carry the definition and then $N_i^1 \subseteq \bigcup_{j<\delta} N_j^0$ and

$\langle N_i^0 : i < \delta \rangle, p_1 \restriction \bigcup_{i<\delta} N_i^0$ contradicts 6.3.

<u>Case II</u>: $\mathrm{cf}(\delta) > \chi$.
   Now by 3.3

   (*) for some $N^* \leq_\mathfrak{K} M_\delta$ of cardinality χ we have p_1 does not χ-split over N^*.

As $\delta = \mathrm{cf}(\delta) \geq \mu > \chi$, for some $i(*) < \delta$ we have $N^* \leq_\mathfrak{K} M_{i(*)}$. This ends the proof of $\otimes$.
So $i(*), N^*$ are well defined. Without loss of generality N^* is saturated. Let $c \in \mathfrak{C}$ realize p_1. We can find a $\leq_\mathfrak{K}$-embedding h of $EM_{\tau(\mathfrak{K})}(\mu^+ + \mu^+, \Phi)$ into $\mathfrak{C}$ such that

   (a) N^* is the h-image of $EM_{\tau(\mathfrak{K})}(\chi, \Phi)$
   (b) h maps $EM_{\tau(\mathfrak{K})}(\mu^+, \Phi)$ onto $M' \leq_\mathfrak{K} M_{i(*)}$
   (c) every $d \in N$ and c belong to the range of h.

By renaming, h is the identity, clearly for some $\alpha < \mu^+$ we have $N \cup \{c\} \subseteq EM_\tau(\alpha \cup [\mu^+, \mu^+ + \alpha))$, so some list $\bar{b}$ of the members of N is $\bar{\sigma}(\ldots, \bar{a}_i, \ldots, a_{\mu^+ + j})_{i < \alpha, j < \alpha}$ (assume $\alpha > \mu$ for simplicity) and $c = \sigma^*(\ldots, \bar{a}_i, \ldots, a_{\mu^+ + j}, \ldots)_{i \in u, j \in w}$ ($u, w \subseteq \mu^+$ finite as, of course, only finitely many $\bar{a}_i$'s are needed for the term σ^*).

For each $\gamma < \mu^+$ we define $\bar{b}^\gamma = \bar{\sigma}(\ldots, \bar{a}_i, \ldots, a_{(1+\gamma)\alpha + j}, \ldots)_{i < \alpha, j < \alpha}$ and $c^\gamma = \sigma^*(\ldots, \bar{a}_i, \ldots, a_{(1+\gamma)\alpha + j}, \ldots)_{i, j}$ and stipulate $\bar{b}^{\mu^+} = \bar{b}, c^{\mu^+} = c$ and let $q = \text{tp}(\bar{b}\hat{\,}c, N^*, \mathfrak{C})$. Clearly $\langle \bar{b}^\gamma \hat{\,} c^\gamma : \gamma < \mu^+ \rangle \hat{\,} \bar{b}\hat{\,}c$ is a strictly indiscernible sequence over N^* and $\subseteq M_\delta \cup \{c\}$, so also $\{\bar{b}^\gamma : \gamma \leq \mu\} \subseteq M_\delta$ is strictly indiscernible over N. [Why? Use $I \supseteq \mu^+ + \mu^+$ which is a strongly μ^{++} saturated dense linear order and use automorphisms of $EM(I, \Phi)$ induced by an automorphism of I.] As c realizes p_1 clearly $\text{tp}(c, M_\delta)$ does not χ-split over N^* hence by 9.3 necessarily $\text{tp}(\bar{b}^\gamma \hat{\,} c, N^*, \mathfrak{C})$ is the same for all $\gamma \leq \mu^+$, hence $\gamma < \mu^+ \Rightarrow \text{tp}(\bar{b}^\gamma \hat{\,} c^{\mu^+}, N^*, \mathfrak{C}) = q$, so by the indiscernibility $\beta < \gamma \leq \mu^+ \Rightarrow \text{tp}(\bar{b}^\beta \hat{\,} c^\gamma, N^*, \mathfrak{C}) = q$.

Similarly for some q_1,

$$\beta < \gamma \leq \mu^+ \Rightarrow \text{tp}(\bar{b}^\gamma \hat{\,} c^\beta, N^*, \mathfrak{C}) = q_1.$$

If $q \neq q_1$, then $\text{tp}(c_0, \bar{b}^1, \mathfrak{C}) \neq \text{tp}(c_2, \bar{b}^1, \mathfrak{C})$, but $\text{Rang}(\bar{b}^\gamma)$ is a model $N^*_\gamma \leq_{\mathfrak{K}} M_{i(*)}, N^* \leq_{\mathfrak{K}} N^*_\gamma$, so by 8.9, for some $v \subseteq \ell g(\bar{b}^\gamma)$ of cardinality χ, $\text{tp}(c_0, \bar{b}^1 \upharpoonright v, \mathfrak{C}) \neq \text{tp}(c_2, \bar{b}^1 \upharpoonright v, \mathfrak{C})$. So clearly we get the $(\chi, \chi, 1)$-order property contradiction to 4.15.
Hence necessarily $\beta \leq \mu^+$ & $\gamma \leq \mu^+$ & $\beta \neq \gamma \Rightarrow \text{tp}(\bar{b}^\beta \hat{\,} c^\gamma, N^*, \mathfrak{C}) = q$. For $\beta = \mu^+, \gamma = 0$ we get that $c^\gamma \in M_{i(*)} \leq_K M_\delta$ realizes $\text{tp}(c^\gamma, N, \mathfrak{C}) = p_1 \upharpoonright N$ as desired. $\square_{9.5}$

We could have proved earlier

9.6 Observation. 1) If M is θ-saturated, $\theta > LS(\mathfrak{K})$ and $\theta < \lambda$ and $N \leq_{\mathfrak{K}} M, N \in K_{\leq \theta}$ then there is $N', N \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} M, N' \in K_\theta$ and every $p \in \mathscr{S}(N)$ realized in M is realized in N'.
2) Assume $\langle N_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasingly continuous, $\delta < \theta^+$ is divisible by $\theta, N_i \in K_{\leq \theta}, N_i \leq_{\mathfrak{K}} M, M$ is θ-saturated, and every $p \in \mathscr{S}(N_i)$ realized in M is realized in N_{i+1} then

    (a) if $\delta = \theta \times \sigma, LS(\mathfrak{K}) < \sigma = \text{cf}(\sigma) \leq \theta$, then N_δ is σ-saturated
    (b) if $\delta = \theta \times \theta, \theta > LS(\mathfrak{K})$, then N_δ is saturated.

9.7 Theorem. *(The Downward Los theorem for λ successors).*
    If λ is successor $\geq \mu(\chi(\Phi^*)) = \mu < \chi < \lambda$, then K is categorical in χ.

9.8 Remark. 1) We intend to try to prove in future work that also in $K_{>\lambda}$ we have categoricity and deal with λ not successor. This calls for using $\mathscr{P}^-(n)$-diagrams as in [Sh 87a], [Sh 87b], etc.
2) We need some theory of orthogonality and regular types parallel to [Sh:a, Ch.V] = [Sh:c, Ch.V], as done in [Sh:h] and then [MaSh 285] (which appeared) and then

(without the upward categoricity) [KlSh 362], [Sh 472]. Then the categoricity can be proved as in those papers.

Proof. Let $K' = \{M \in K : M$ is $\chi(\Phi^*)$-saturated of cardinality $\geq \chi(\Phi^*)\}$. So

$(*)_0$ there is $M \in K_\lambda$ which is λ-saturated
[why? by 2.6, 1.7, as λ is regular]

$(*)_1$ K' is closed under $\leq_{\mathfrak{K}}$-increasing unions

$(*)_2$ if $\chi \geq \beth_{(2^{LS(\mathfrak{K})})^+}(\chi(\Phi^*))$ and $M \in K_\chi$ <u>then</u> $M \in K'_\chi$, moreover M is $\beth_{(2^{LS(\mathfrak{K})})^+}(\chi(\Phi^*))$-saturated
[Why? Otherwise by 8.5 there is a non $LS(\mathfrak{K})^+$-saturated $M \in K_\lambda$ contradicting $(*)_0$, or use 8.10. For the "Moreover" use 8.6 instead of 8.5]

$(*)_3$ if $M \in K', p \in \mathscr{S}(M)$ <u>then</u> for some $M_0 \leq_{\mathfrak{K}} M, M_0 \in K'_{\chi(\Phi^*)}$ and p does not $\chi(\Phi^*)$-split over M_0
[why? by 3.3, 1.7]

$(*)_4$ <u>Definition</u>: for $\chi \in [\chi(\Phi^*), \lambda)$ and $M \in K'_\chi$ and $p \in \mathscr{S}(M)$ we say p is minimal if

  (a) p is not algebraic which means p is not realized by any $c \in M$

  (b) if $M \leq_{\mathfrak{K}} M' \in K'_\chi$ and $p_1, p_2 \in \mathscr{S}(M')$ are non-algebraic extending p, <u>then</u> $p_1 = p_2$

$(*)_5$ <u>Fact</u>: if $M \in K'_\chi$ is saturated, $\chi \in [\chi(\Phi^*), \lambda)$, <u>then</u> some $p \in \mathscr{S}(M)$ is minimal
[Why? If not, we choose by induction on $\alpha \leq \chi$ for every $\eta \in {}^\alpha 2$ and triple (M_η, N_η, a_η) and $h_{\eta, \eta \restriction \beta}$ for $\beta \leq \alpha$ such that:

  (a) $M_\eta <_{\mathfrak{K}} N_\eta$ and $a_\eta \in N_\eta \backslash M_\eta$

  (b) $\langle M_{\eta \restriction \beta} : \beta \leq \alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasingly continuous

  (c) $M_{\eta \restriction \beta} <^1_{\mu, \omega} M_{\eta \restriction (\beta+1)}$

  (d) $h_{\eta, \eta \restriction \beta}$ is a $\leq_{\mathfrak{K}}$- embedding of $N_{\eta \restriction \beta}$ into N_η which is the identity on $M_{\eta \restriction \beta}$ and maps $a_{\eta \restriction \beta}$ to a_η

  (e) if $\gamma \leq \beta \leq \alpha, \eta \in {}^\alpha 2$, then $h_{\eta, \eta \restriction \gamma} = h_{\eta, \eta \restriction \beta} \circ h_{\eta \restriction \beta, \eta \restriction \gamma}$

  (f) $M_{\eta^\frown \langle 0 \rangle} = M_{\eta^\frown \langle 1 \rangle}$ but $\text{tp}(a_{\eta^\frown \langle 0 \rangle}, M_{\eta^\frown \langle 0 \rangle}, N_{\eta^\frown \langle 0 \rangle}) \neq \text{tp}(a_{\eta^\frown \langle 1 \rangle}, M_{\eta^\frown \langle 1 \rangle}, N_{\eta^\frown \langle 1 \rangle})$

  (g) $M_\eta <_{\mathfrak{K}} \mathfrak{C}$.

No problem to carry the definition and let $\kappa = \text{Min}\{\kappa : 2^\kappa > \chi\}$ and choose $M <_{\mathfrak{K}} \mathfrak{C}, \|M\| \leq \chi, \eta \in {}^{\kappa >}2 \Rightarrow M_\eta \subseteq M$ hence $\eta \in {}^\kappa 2 \Rightarrow M_\eta \subseteq M$ so $\{\text{tp}(a_\eta, M, \mathfrak{C}) : \eta \in {}^\kappa 2\}$ is a subset of $\mathscr{S}(M)$ of cardinality $2^\kappa > \chi$. So then we can get a contradiction to stability in χ].

$(*)_6$ Fix $M^* \in K'_{\chi(\Phi^*)}$ and minimal $p^* \in \mathscr{S}(M^*)$

$(*)_7$ if $M^* \leq_{\mathfrak{K}} M \in K'_{<\lambda}$, <u>then</u> p^* has a non-algebraic extension $p \in \mathscr{S}(M)$, moreover, if M is saturated, it is unique and also p is minimal
[Why? Existence by 6.3, uniqueness modulo $E_{\chi(\Phi^*)}$ follows hence uniqueness by locality lemma 9.3. Applying this to a saturated extension M' of M of cardinality $\|M\|$ we get p is minimal].

Let λ_1 be the predecessor of λ.

$(*)_8$ there are no M_1, M_2 such that:

    (a) $M^* \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$

    (b) M_1, M_2 are saturated of cardinality λ_1

    (c) $M_1 \neq M_2$

    (d) no $c \in M_2 \backslash M_1$ realizes p^*

    [Why? If there are, we choose by induction on $\zeta < \lambda, N_\zeta \in K_{\lambda_1}$ is $\leq_{\mathfrak{K}}$-increasingly continuous, each N_ζ is saturated, $N_0 = M_1, N_\zeta \neq N_{\zeta+1}$ and no $c \in N_{\zeta+1} \backslash N_\zeta$ realizes p^*. If we succeed, then $N = \bigcup_{\zeta < \lambda} N_\zeta$ is in K_λ (as $N_\zeta \neq N_{\zeta+1}$!) but no $c \in N \backslash N_\zeta$ realizes p^*
(why? as $\{\zeta : c \notin N_\zeta\}$ is an initial segment of λ, non-empty as 0 is in so it has a last element ζ, so $c \in N_{\zeta+1} \backslash N_\zeta$ so realizes p^*, contradiction); hence N is not saturated, contradiction. For $\zeta = 0, N_0 = M_1$ is okay by clause (b). If ζ is limit $< \lambda$, let $N_\zeta = \bigcup_{\varepsilon < \zeta} N_\varepsilon$, clearly $N_\zeta \in K_{\lambda_1}$ and it is saturated by 9.5. If $\zeta = \varepsilon + 1$, note that as N_ε, M_1 are saturated and in K_{λ_1} and $\leq_{\mathfrak{K}}$-extends M^* which has smaller cardinality, there is an isomorphism f_ζ from M_1 onto N_ε which is the identity on M^*. We define N_ζ such that there is an isomorphism f_ζ^+ from M_2 onto N_ζ extending f_ζ. By assumption (b), $N_\zeta \in K_{\lambda_1}$ is saturated by assumption (c), $N_\zeta \neq N_{\zeta+1}$, and by assumption (d), no $c \in N_{\zeta+1} \backslash N_\zeta$ realizes p^* (as $f \upharpoonright M^* =$ the identity). So as said above, we have derived the desired contradiction].

$(*)_9$ if $M \in K'_{<\lambda}$ and $M^* \leq M <_{\mathfrak{K}} N, M$ has cardinality $\geq \theta^* = \beth_{(2^{\chi(\Phi^*)})^+}$, <u>then</u> some $c \in N \backslash M$ realizes p^*.
[Why? By $(*)_2$, M, N are θ^*-saturated. So we can find saturated $M' \leq_{\mathfrak{K}} M, N' \leq_{\mathfrak{K}} N$ of cardinality θ^* such that $M' = N' \cap M, M^* \neq N'$ (why? by observation 9.6). So still no $c \in N' \backslash M'$ realizes p^*. We would like to transfer the appropriate omitting type theorem of this situation from θ^* to λ_1; the least trivial point is preserving the saturation. But this can be expressed as: "is isomorphic to $EM(I, \Phi)$ for some linear order I" for appropriate Φ, and this is easily transferred].

$(*)_{10}$ if $M \in K'_{\leq\lambda}$ has cardinality $\geq \theta^* = \beth_{(2^{\chi(\Phi^*)})^+}$ then it is θ^*-saturated (so $\in K'_{\leq\lambda}$)
[why? included in the proof of $(*)_9$]

$(*)_{11}$ if $M \in K'_{\leq\lambda}$ has cardinality $\geq \theta^*$, <u>then</u> M is saturated
[why? Assume not; by $(*)_{10}$, M is θ^*-saturated let θ be such that M is θ-saturated but not θ^+-saturated; by $(*)_{10}$, $\theta \geq \theta^*$, without loss of generality $M^* \leq_{\mathfrak{K}} M$. Let $M_0 \leq_{\mathfrak{K}} M$ be such that $M_0 \in K_\theta$ and some $q \in \mathscr{S}(M_0)$ is omitted by M and without loss of generality $M^* \leq_{\mathfrak{K}} M_0$.
Now choose by induction on $i < \theta^+$ a triple (N_i^0, N_i^1, f_i) such that:

    (a) $N_i^0 \leq_{\mathfrak{K}} N_i^1$ belong to K_θ and are saturated

    (b) N_i^0 is $\leq_{\mathfrak{K}}$-increasingly continuous

    (c) N_i^1 is $\leq_{\mathfrak{K}}$-increasingly continuous

    (d) $N_0^0 = M_0$ and $d \in N_0^1$ realizes q

(e) f_i is a $\leq_\mathfrak{K}$-embedding of N_i^0 into M and $f_0 = \mathrm{id}_{M_0}$

(f) for each i, for some $c_i \in N_i^1 \backslash N_i^0$ we have $c_i \in N_{i+1}^0$

(g) f_i is increasing continuous.

> If we succeed, let $E = \{\delta < \theta^+ : \delta \text{ limit and for every } i < \delta \text{ and } c \in N_i^1$
> we have $(\exists j < \theta^+)(c_j = c) \to (\exists j < \delta)(c_j = c)\}$. Clearly E is a club
> of θ^+, and for each $\delta \in E, c_\delta$ belongs to $N_\delta^1 = \bigcup_{i < \delta} N_i^1$ so there is $i < \delta$
> such that $c_\delta \in N_i^1$, so for some $j < \delta, c = c_j$ so $c_\delta = c_j \in N_{j+1}^0 \leq_\mathfrak{K} N_\delta^0$,
> contradiction to clause (f).
>
> So we are stuck for some ζ, now $\zeta \neq 0$ trivially. Also ζ not limit
> by 9.5, so $\zeta = \varepsilon + 1$. Now if $N_\varepsilon^0 = N_\varepsilon^1$, then $f_\varepsilon(d) \in M$ realizes q
> a contradiction, so $N_\varepsilon^0 <_\mathfrak{K} N_\varepsilon^1$. Also $f_\varepsilon(N_\varepsilon^0) <_\mathfrak{K} M$ by cardinality
> consideration. Now by $(*)_9$ some $c_\varepsilon \in N_\varepsilon^1 \backslash N_\varepsilon^0$ realizes p^*.
>
> We can find $N_\zeta' \leq_\mathfrak{K} M$ such that $f_\varepsilon(N_\varepsilon^0) <_\mathfrak{K} N_\zeta' \in K_\theta, N_\zeta'$ saturated
> (why? by 9.6).
>
> Again by $(*)_9$ we can find $c_\zeta' \in N_\zeta' \backslash f_\varepsilon(N_\varepsilon^0)$ realizing p^*. By $(*)_5$ clearly
> $\mathrm{tp}(c_\varepsilon', f_\varepsilon(N_\varepsilon^0), M) = f_\varepsilon(\mathrm{tp}(c_\varepsilon, N_\varepsilon^0, N_\varepsilon^1))$ so we can find $N_\zeta^1 \in K_\theta$ which
> is a $\leq_\mathfrak{K}$-extension of N_ε^1 and a $\leq_\mathfrak{K}$-embedding g_ε of N_ζ' into N_ζ^1 which
> extends f_ε^{-1} and maps c_ε' to c_ε. Without loss of generality N_ζ^1 is
> saturated. Let $N_\zeta^0 = g_\varepsilon(N_\zeta')$ and N_ζ^1, c_ε were already defined. So we
> can carry the construction, contradiction, so $(*)_{11}$ holds].

$(*)_{12}$ K_λ is categorical in every $\chi \in [\beth_{(2^{\chi(\Phi^*)})^+}, \lambda)$

> [why? by $(*)_{11}$ every model is saturated and the saturated model is unique].
>
> $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square_{9.7}$

REFERENCES.

[GrSh 238] Rami Grossberg and Saharon Shelah. A nonstructure theorem for an infinitary theory which has the unsuperstability property. *Illinois Journal of Mathematics*, **30**:364–390, 1986. Volume dedicated to the memory of W.W. Boone; ed. Appel, K., Higman, G., Robinson, D. and Jockush, C.

[J] Thomas Jech. *Set theory.* Academic Press, New York, 1978.

[KlSh 362] Oren Kolman and Saharon Shelah. Categoricity of Theories in $L_{\kappa,\omega}$, when κ is a measurable cardinal. Part 1. *Fundamenta Mathematicae*, **151**:209–240, 1996.

[MaSh 285] Michael Makkai and Saharon Shelah. Categoricity of theories in $L_{\kappa\omega}$, with κ a compact cardinal. *Annals of Pure and Applied Logic*, **47**:41–97, 1990.

[Sh 600] Saharon Shelah. Categoricity in abstract elementary classes: going up inductive step. *in preparation.*

[Sh 576] Saharon Shelah. Categoricity of an abstract elementary class in two successive cardinals. *Israel Journal of Mathematics*, **submitted**.

[Sh 472] Saharon Shelah. Categoricity of Theories in $L_{\kappa^*\omega}$, when κ^* is a measurable cardinal. Part II. *Fundamenta Mathematicae*, **submitted**.

[Sh:e] Saharon Shelah. *Non–structure theory*, accepted. Oxford University Press.

[Sh 620] Saharon Shelah. Special Subsets of $^{\mathrm{cf}(\mu)}\mu$, Boolean Algebras and Maharam measure Algebras. *General Topology and its Applications - Proc. of Prague Topological Symposium 1996*, **accepted**.

[Sh:h] Saharon Shelah. *Universal classes*, in preparation.

[Sh 3] Saharon Shelah. Finite diagrams stable in power. *Annals of Mathematical Logic*, **2**:69–118, 1970.

[Sh 48] Saharon Shelah. Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$. *Israel Journal of Mathematics*, **20**:127–148, 1975.

[Sh:a] Saharon Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, xvi+544 pp, $62.25, 1978.

[Sh 87a] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A. *Israel Journal of Mathematics*, **46**:212–240, 1983.

[Sh 87b] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B. *Israel Journal of Mathematics*, **46**:241–273, 1983.

[Sh 88] Saharon Shelah. Classification of nonelementary classes. II. Abstract elementary classes. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Mathematics*, pages 419–497. Springer,

Berlin, 1987. Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.

[Sh 220]   Saharon Shelah. Existence of many $L_{\infty,\lambda}$-equivalent, nonisomorphic models of T of power λ. *Annals of Pure and Applied Logic*, **34**:291–310, 1987. Proceedings of the Model Theory Conference, Trento, June 1986.

[Sh 300]   Saharon Shelah. Universal classes. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Mathematics*, pages 264–418. Springer, Berlin, 1987. Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.

[Sh:c]   Saharon Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, xxxiv+705 pp, 1990.

[ShVi 635] Saharon Shelah and Andrés Villaveces. Toward Categoricity for Classes with no Maximal Models. *Annals of Pure and Applied Logic*, **accepted**.